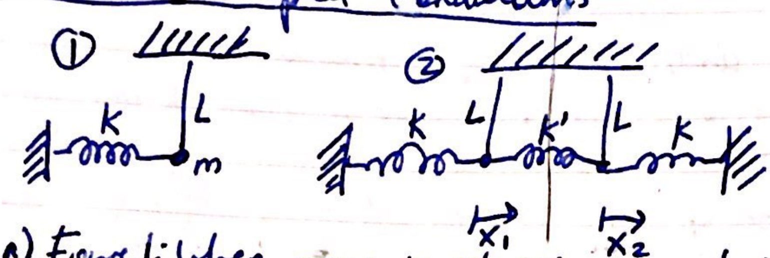
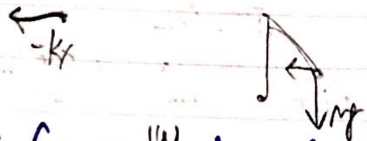
12/30/19 8.03 pset 4

Problem 1: Coupled Pendulums



a) Figure 1: When spring is relaxed, the rod is vertical.
What is the frequency of small oscillations of mass in plane of rod & spring.



$mg\sin(\theta) = -mg\dfrac{x}{L}$ $ma = F_x = -kx - mg\dfrac{x}{L}$ $\ddot{x} + \left[\dfrac{k}{m} + \dfrac{g}{L}\right]x$

$m\ddot{x} + kx + \dfrac{mg}{L}x = 0$ $\boxed{\omega_0 = \sqrt{\dfrac{k}{m} + \dfrac{g}{L}}}$

b) Find frequencies (ω) & amplitude ratios for the two normal modes of the system. (cont'd)

normal mode: all parts of the system move w/ same frequency

1) If $x_2 = x_1$, K' doesn't stretch. Therefore:
$\omega_1^2 = \dfrac{k}{m} + \dfrac{g}{L}$

2) $x_2 = -x_1$
$\Rightarrow \omega_2^2 = \left(\dfrac{k+2k'}{m} + \dfrac{g}{L}\right)$

$F(K' \text{spring}) = -k'(2x_1)$?

normal: $m\ddot{x}_1 = -kx_1 - \underbrace{k'(x_1 - x_2)}_{} - mg\dfrac{x_1}{L}$
$= k'(x_1 - (-x_1))$
$= k'2(2x_1)$

longer way

$m\ddot{x}_1 = -kx_1 - k'(x_1 - x_2) - mg\dfrac{x_1}{L}$ $m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1) - mg\dfrac{x_2}{L}$

☆ $\Rightarrow \ddot{x}_1 + \left(\dfrac{k+k'}{m} + \dfrac{g}{L}\right)x_1 - \dfrac{k'}{m}x_2 = 0$ $\ddot{x}_2 + \left(\dfrac{k+k'}{m} + \dfrac{g}{L}\right)x_2 - \dfrac{k'}{m}x_1 = 0$ ☆ Equations

$x_1 = C_1\cos(\omega t + \phi) = C_1 e^{i\omega t}$ $x_2 = C_2\cos(\omega t + \phi) = C_2 e^{i\omega t}$

plug in $x_1 = C_1 e^{i\omega t}$, $x_2 = C_2 e^{i\omega t}$ $\Rightarrow$

$C_1(-\omega^2)\cancel{e^{i\omega t}} + \left(\dfrac{k+k'}{m} + \dfrac{g}{L}\right)C_1\cancel{e^{i\omega t}} - \dfrac{k'}{m}C_2\cancel{e^{i\omega t}} = 0$

$\Rightarrow \left(\dfrac{k+k'}{m} + \dfrac{g}{L} - \omega^2\right)C_1 - \dfrac{k'}{m}C_2 = 0$

☆ Set situation up as an eigenvalue problem

$C_2 - \omega^2\cancel{e^{i\omega t}} + \left(\dfrac{k+k'}{m} + \dfrac{g}{L}\right)C_2\cancel{e^{i\omega t}} - \dfrac{k'}{m}C_1\cancel{e^{i\omega t}} = 0$

$-\left(\dfrac{k+k'}{m} + \dfrac{g}{L} - \omega^2\right)C_2 - \dfrac{k'}{m}C_1 = 0$

$$\begin{bmatrix} \dfrac{k+k'}{m} + \dfrac{g}{L} - \omega^2 & -\dfrac{k'}{m} \\ -\dfrac{k'}{m} & \dfrac{k+k'}{m} + \dfrac{g}{L} - \omega^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\left(\dfrac{k+k'}{m} + \dfrac{g}{L} - \omega^2\right)^2 - \left(\dfrac{k'}{m}\right)^2 \Rightarrow \dfrac{k+k'}{m} + \dfrac{g}{L} - \omega^2 = \pm\dfrac{k'}{m}$ $\omega^2 = \dfrac{k+k'}{m} + \dfrac{g}{L} \pm \dfrac{k'}{m}$

$\boxed{\omega_1^2 = \dfrac{k}{m} + \dfrac{g}{L} \quad \omega_2^2 = \dfrac{k+2k'}{m} + \dfrac{g}{L}}$

$\rightarrow = \dfrac{k'}{m}C_1 = \dfrac{k'}{m}C_2 \Rightarrow \boxed{C_1 = C_2} \Rightarrow x_1 = x_2$

$\rightarrow = -\dfrac{k'}{m}C_1 - \dfrac{k'}{m}C_2 = 0$ $\boxed{C_1 = -C_2} \Rightarrow x_2 = -x_1$

this process is how to find normal modes, but not how to solve

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c) At $t=0$, the system is released from rest ($\dot{x}_1=0$, $\dot{x}_2=0$) with $x_1=d$ and $x_2=0$. Find x_1 and x_2 as a function of time.

– For solving, try adding up eq'ns

$$\ddot{x}_1+\left(\frac{k+k'}{m}+\frac{g}{L}\right)x_1-\frac{k'}{m}x_2=0 \qquad \ddot{x}_2+\left(\frac{k+k'}{m}+\frac{g}{L}\right)x_2-\frac{k'}{m}x_1=0$$

addition: $\frac{d^2}{dt^2}(x_1+x_2)+\left(\frac{k}{m}+\frac{g}{L}\right)x_1+\left(\frac{k}{m}+\frac{g}{L}\right)x_2=0$

$$=\frac{d^2}{dt^2}(x_1+x_2)+\left(\frac{k}{m}+\frac{g}{L}\right)(x_1+x_2)=0 \qquad \omega_1^2=\frac{k}{m}+\frac{g}{L}$$

subtraction: $\frac{d^2}{dt^2}(x_1-x_2)+\left(\frac{k+2k'}{m}+\frac{g}{L}\right)x_1-\left(\frac{k+2k'}{m}+\frac{g}{L}\right)x_2$

irrelevant to part C!

$$=\frac{d^2}{dt^2}(x_1-x_2)+\left(\frac{k+2k'}{m}+\frac{g}{L}\right)(x_1-x_2) \qquad \omega_2^2=\frac{k+2k'}{m}+\frac{g}{L}$$

$$\left.\begin{aligned}x_1&=A\cos(\omega_A t+\phi_A)+B\cos(\omega_B t+\phi_B)\\ x_2&=A\cos(\omega_A t+\phi_A)-B\cos(\omega_B t+\phi_B)\end{aligned}\right]$$

remember amplitude ratios: $C_1=C_2$ & $C_1=-C_2$? this is the general sol'n containing both.

$\dot{x}_1(0)=-A\omega_A\sin(\phi_A)-B\omega_B\sin(\phi_B)=0$

$\dot{x}_2(0)=-A\omega_A\sin(\phi_A)+B\omega_B\sin(\phi_B)=0$

$\dot{x}_1(0)+\dot{x}_2(0)=0=-2A\omega_A\sin(\phi_A)=0$

$\dot{x}_1(0)-\dot{x}_2(0)=0=-2B\omega_B\sin(\phi_B)=0$

$$\boxed{\begin{aligned}\phi_A&=0\\ \phi_B&=0\end{aligned}}$$

$x_1(0)=d \quad x_2(0)=0.\quad x_1(0)=A\cos(0)+B\cos(0)=d=A+B$

$x_2(0)=A\cos(0)-B\cos(0)=0=A-B \Rightarrow 2A=d \quad A=\frac{d}{2}$

$2B=d \quad B=\frac{d}{2}$

$$\boxed{\begin{aligned}x_1&=\frac{d}{2}\cos(\omega_A t)+\frac{d}{2}\cos(\omega_B t)\\ x_2&=\frac{d}{2}\cos(\omega_A t)-\frac{d}{2}\cos(\omega_B t)\end{aligned}}$$

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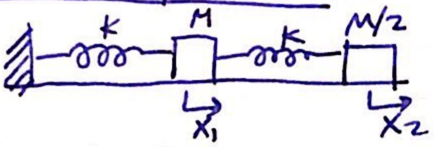
Problem 2: Coupled Blocks



random question:
why are the forces acting in different directions yet have same sign? Because if you compress a spring it also exerts a force...

a) Find the coupled differential equations for $X_1(t)$ and $X_2(t)$. Express results in terms of characteristic frequency $\omega_0=\sqrt{K/M}$

$$M\ddot{X}_1 = -KX_1 - K(X_1-X_2) \qquad \frac{M}{2}\ddot{X}_2 = -K(X_2-X_1)$$

$$\ddot{X}_1 + \frac{K}{M}X_1 + \frac{K}{M}(X_1-X_2) \qquad \ddot{X}_2 = -\frac{2}{M}K(X_2-X_1)$$

$$= \ddot{X}_1 + \frac{2K}{M}X_1 - \frac{K}{M}X_2 \qquad \ddot{X}_2 + \frac{2}{M}KX_2 - \frac{2}{M}KX_1$$

$$= \boxed{\ddot{X}_1 + 2\omega_0^2X_1 - \omega_0^2X_2 = 0} \qquad \boxed{\ddot{X}_2 + 2\omega_0^2X_2 - 2\omega_0^2X_1 = 0}$$

b) Find the frequencies of normal modes of vibrations of system in terms of ω_0

sub $X_1=C_1e^{i\omega t}$ $X_2=C_2e^{i\omega t}$

$$\Rightarrow 0 = -C_1\omega^2\cancel{e^{i\omega t}} + 2\omega_0^2C_1\cancel{e^{i\omega t}} - \omega_0^2C_2\cancel{e^{i\omega t}}$$
$$= -C_1\omega^2 + 2\omega_0^2C_1 - \omega_0^2C_2$$
$$= C_1(2\omega_0^2-\omega^2) - C_2(\omega_0^2)$$

$$-C_2\omega^2\cancel{e^{i\omega t}} + 2\omega_0^2C_2\cancel{e^{i\omega t}} - 2\omega_0^2C_1\cancel{e^{i\omega t}} = 0$$
$$= C_2(2\omega_0^2-\omega^2) - C_1 2\omega_0^2 = 0$$

$$\begin{bmatrix} 2\omega_0^2-\omega^2 & -\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2-\omega^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(2\omega_0^2-\omega^2)^2 - (2\omega_0^4) = 0 \Rightarrow 2\omega_0^2-\omega^2 = \pm\sqrt{2}\omega_0^2 \Rightarrow \boxed{\omega^2 = (2\pm\sqrt{2})\omega_0^2}$$

c) Find the amplitude ratios for each of the normal modes.

basically find C_1 & C_2.

$$\rightarrow C_1(2\omega_0^2 - (2+\sqrt{2})\omega_0^2) - C_2(\omega_0^2) \qquad C_1(2\omega_0^2 - (2-\sqrt{2})\omega_0^2) - C_2(\omega_0^2)$$

$$0 = C_1(-\sqrt{2}\omega_0^2) - C_2(\omega_0^2) \qquad 0 = C_1(\sqrt{2}\omega_0^2) - C_2\omega_0^2$$

$$-C_1\sqrt{2}\cancel{\omega_0^2} = C_2\cancel{\omega_0^2} \quad \boxed{-\sqrt{2}C_1 = C_2} \qquad \sqrt{2}C_1\cancel{\omega_0^2} = C_2\cancel{\omega_0^2} \quad \boxed{\sqrt{2}C_1 = C_2}$$

(You can do this w/ the other eq'n if you really wanted to check)

d) At $t=0$, the lighter block is given an initial velocity of $-V_0$ [$X_1(0)=0, X_2(0)=0, \dot{X}_1(0)=0, \dot{X}_2(0)=-V_0$]. Find X_1 & X_2 as functions of time.

$$X_1(t) = A\cos(\omega_At+\phi_A) + B\cos(\omega_Bt+\phi_B)$$
$$X_2(t) = -\sqrt{2}A\cos(\omega_At+\phi_A) + \sqrt{2}B\cos(\omega_Bt+\phi_B)$$

$$2\sqrt{2}B\omega_B(-1) = -V_0$$

$$\dot{X}_1(0) = -\omega_AA\sin(\phi_A) - \omega_BB\sin(\phi_B) = 0 \qquad \sqrt{2}\dot{X}_1(0) + \dot{X}_2(0) = 2\sqrt{2}B\omega_B\sin(\phi_B) = -V_0$$

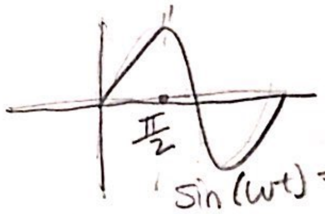
$$\dot{X}_2(0) = \sqrt{2}A\omega_A\sin(\phi_A) - \sqrt{2}B\omega_B\sin(\phi_B) = -V_0 \qquad \sqrt{2}\dot{X}_1(0) - \dot{X}_2(0) = -2\sqrt{2}\omega_AA\sin(\phi_A) = V_0$$

$$-2\sqrt{2}\omega_AA = V_0$$

$$X_1(0) = A\cos(\phi_A) + B\cos(\phi_B) = 0 \qquad \sqrt{2}X_1(0) + X_2(0) = 2\sqrt{2}B\cos(\phi_B) = 0 \Rightarrow \phi_B = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

$$X_2(0) = -\sqrt{2}A\cos(\phi_A) + \sqrt{2}B\cos(\phi_B) = 0 \qquad \sqrt{2}X_1(0) - X_2(0) = 2\sqrt{2}A\cos(\phi_A) = 0 \Rightarrow \phi_A = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

$\Rightarrow -2\sqrt{2}\,B\omega_B = -V_0 \quad -2\sqrt{2}\,\omega_A A = V_0 \quad \Rightarrow A = -\frac{V_0}{2\sqrt{2}\,\omega_A} \quad B = \frac{V_0}{2\sqrt{2}\,\omega_B}$



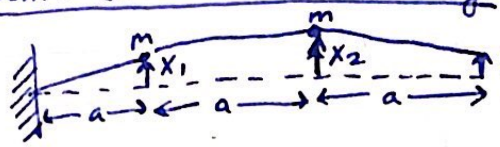
$$x_1(t) = -\frac{V_0}{2\sqrt{2}\,\omega_A}\cos\left(\omega_A t - \frac{\pi}{2}\right) + \frac{V_0}{2\sqrt{2}\,\omega_B}\cos\left(\omega_B t - \frac{\pi}{2}\right)$$

$\sin(\omega t) = \cos(\omega t - \frac{\pi}{2})$

$$\boxed{x_1(t) = -\frac{V_0}{2\sqrt{2}\,\omega_A}\sin(\omega_A t) + \frac{V_0}{2\sqrt{2}\,\omega_B}\sin(\omega_B t)}$$

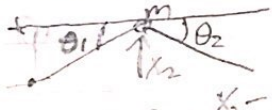
$$\boxed{x_2(t) = \frac{V_0}{2\omega_A}\sin(\omega_A t) + \frac{V_0}{2\omega_B}\sin(\omega_B t)}$$
$\leftarrow$ different from soln manual but I think it's fine?

Problem 3: Driven Beads on a String



- System is on a frictionless horizontal surface
- String under tension T at all times.
- one end of string is fixed. other goes through transverse displacement $d(t) = \Delta\cos(\omega t)$
- beads execute small amplitude horizontal motion perpendicular to string.

- Find steady state response of the beads $x_1(t)$ and $x_2(t)$.
- Make a sketch of the amplitude as a function of frequency.



$\sin\theta_1 \approx \frac{x_2 - x_1}{a}$

$\sin\theta_2 \approx \frac{x_2 - d(t)}{a}$

$m\ddot{x}_2 = -T\sin\theta_1 - T\sin\theta_2$

or $m\ddot{x}_2 + T(\sin\theta_1 + \sin\theta_2) = 0$

$m\ddot{x}_2 + T\left(\frac{x_2 - x_1}{a} + \frac{x_2 - d(t)}{a}\right) = 0$

$= m\ddot{x}_2 a + T(x_2 - x_1) + T(x_2 - d(t))$

$= \frac{m\ddot{x}_2 a}{T} + x_2 - x_1 + x_2 = d(t) = \Delta\cos(\omega t)$

$\Rightarrow \Delta\cos(\omega t) = \frac{ma}{T}\ddot{x}_2 + 2x_2 - x_1$

$\Rightarrow \boxed{\frac{T}{ma}d(t) = \ddot{x}_2 + \frac{2T}{ma}x_2 - \frac{T}{ma}x_1}$

lets you find just one variable

Cramer's Rule:

this is new to me!

i.e. $2x + 1y + 1z = 3$

$1x - 1y - 1z = 0$

$1x + 2y + 1z = 0$

D ⇒ determinant

$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$ answer column: $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3$

$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -6$

$x = \frac{D_x}{D} \quad y = \frac{D_y}{D}$

$m\ddot{x}_1 = -T\sin\theta_1 - T\sin\theta_2$

where $\sin\theta_1 \approx \frac{x_1 - 0}{a}$

$\sin\theta_2 \approx \frac{x_1 - x_2}{a}$

$m\ddot{x}_1 = -T\frac{x_1}{a} - T\left(\frac{x_1 - x_2}{a}\right) = 0 \quad m\ddot{x}_1 + T\left(\frac{x_1}{a} + \frac{x_1 - x_2}{a}\right) = 0 = m\ddot{x}_1 + T\left(\frac{2x_1 - x_2}{a}\right) = 0$

$\Rightarrow \ddot{x}_1 + \frac{T}{m}\frac{2x_1}{a} - \frac{T}{m}\frac{x_2}{a} = 0 = \boxed{\ddot{x}_1 + \frac{2T}{ma}x_1 - \frac{T}{ma}x_2 = 0}$ ω-

Ansatz: $x_1 = C_1 e^{i\omega t} \quad x_2 = C_2 e^{i\omega t}$

$\omega_0^2 = \frac{T}{ma}$

$\left[-C_2\omega^2 + \frac{2T}{ma}C_2 - \frac{T}{ma}C_1\right]e^{i\omega t} = \frac{T}{ma}\Delta\cos(\omega t)$

$\left[-C_1\omega^2 + \frac{2T}{ma}C_1 - \frac{T}{ma}C_2\right]\cancel{e^{i\omega t}} = 0$

$$\begin{bmatrix} \frac{2T}{ma} - \omega^2 & -\frac{T}{ma} \\ -\frac{T}{ma} & \frac{2T}{ma} - \omega^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{T}{ma}\Delta \end{bmatrix} = \begin{bmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_0^2\Delta \end{bmatrix}$$

Cramer's Rule: $\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = (2\omega_0^2 - \omega^2)^2 - \omega_0^4 = \omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4 = (\omega^2 - 3\omega_0^2)(\omega^2 - \omega_0^2)$

$D_{C_1} = \begin{vmatrix} 0 & -\omega_0^2 \\ \omega_0^2\Delta & 2\omega_0^2 - \omega^2 \end{vmatrix}$

$= \omega_0^4\Delta$

$D_{C_2} = \begin{vmatrix} 2\omega_0^2 - \omega^2 & 0 \\ -\omega_0^2 & \omega_0^2\Delta \end{vmatrix}$

$= (2\omega_0^2 - \omega^2)\omega_0^2\Delta$

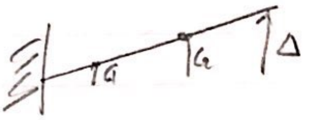
$$\boxed{C_1 = \frac{\omega_0^4\Delta}{(\omega^2 - 3\omega_0^2)(\omega^2 - \omega_0^2)} \qquad C_2 = \frac{(2\omega_0^2 - \omega^2)\omega_0^2\Delta}{(\omega^2 - 3\omega_0^2)(\omega^2 - \omega_0^2)}}$$

sketch of amplitude as a function of frequency

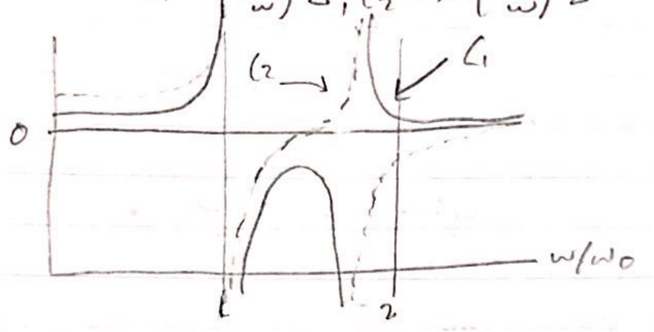
$$C_1 = \frac{\omega_0^4 \Delta}{(\omega^2-\omega_0^2)(\omega^2-3\omega_0^2)} \qquad C_2 = \frac{\omega_0^2\Delta(2\omega_0^2-\omega^2)}{(\omega^2-\omega_0^2)(\omega^2-3\omega_0^2)}$$

$C_2 = 0$ at $\omega = \sqrt{2}\omega_0$

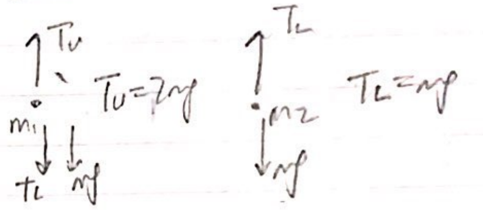
at $\omega = 0$ $\quad C_1 = \frac{1}{3}\Delta,\ C_2 = \frac{2}{3}\Delta$



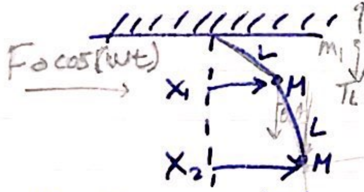
$\omega \to \infty$ $\quad C_1 \to \left(\frac{\omega_0}{\omega}\right)^4\Delta,\ C_2 \to -\left(\frac{\omega_0}{\omega}\right)^2\Delta$



Problem 4 Free Body Diagrams



Problem 4: Driven Coupled Pendulums



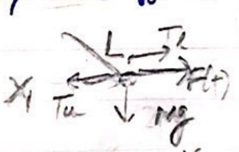
state of pendulum given by $x_1(t)$ & $x_2(t)$

a) What is the tension in the upper string when the pendulum is at rest? $\boxed{2Mg}$ What is the tension in the lower string under same circumstances? $\boxed{Mg}$

For small displacements from equilibrium, the tension in string do not change from equilibrium values. A horizontal force $F_{x_1}(t) = F_0\cos(\omega t)$ is applied to upper of two masses

b) Find coupled differential eqns $x_1(t)$ & $x_2(t)$. Express results in terms of characteristic frequency $\omega_0 = \sqrt{g/L}$

X_1



$$T_u = T_1\cos\theta = 2Mg\frac{X_1}{L}$$

$$T_L = T_2\cos\theta = Mg\frac{X_2-X_1}{L}$$

$$M\ddot{X}_1 = F_0\cos(\omega t) - 2Mg\frac{X_1}{L} + Mg\frac{X_2-X_1}{L}$$

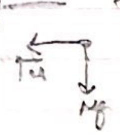
$$M\ddot{X}_2 = -Mg\frac{X_2-X_1}{L}$$

$$\ddot{X}_1 = \frac{F_0}{M}\cos(\omega t) - 2\frac{g}{L}X_1 + \frac{g}{L}(X_2-X_1)$$

$$\ddot{X}_1 = \frac{F_0}{M}\cos(\omega t) - 3\frac{g}{L}X_1 + \frac{g}{L}X_2 \qquad \boxed{\ddot{X}_1 + 3\omega_0^2 X_1 - \omega_0^2 X_2 = \frac{F(t)}{M}}$$

$$\ddot{X}_2 = -\frac{g}{L}X_2 + \frac{g}{L}X_1 \qquad \boxed{\ddot{X}_2 + \omega_0^2 X_2 - \omega_0^2 X_1 = 0}$$

X_2



$$T_u = T_2\cos\theta = Mg\frac{X_2-X_1}{L}$$

c) Find steady state response of two masses $X_1(t)$ & $X_2(t)$.

Make sketch of amplitude as function of frequency for each of the masses

Ansatz: $X_1 = C_1 e^{i\omega t} \quad X_2 = C_2 e^{i\omega t}$

$$[-\omega^2 C_1 + 3\omega_0^2 C_1 - \omega_0^2 C_2]\cancel{e^{i\omega t}} = \frac{F_0}{M}\cancel{\cos(\omega t)}$$

$$[-C_2\omega^2 + \omega_0^2 C_2 - \omega_0^2 C_1]\cancel{e^{i\omega t}} = 0$$

$$\begin{bmatrix} 3\omega_0^2-\omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\omega^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{F_0}{M} \\ 0 \end{bmatrix} \qquad \begin{vmatrix} 3\omega_0^2-\omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\omega^2 \end{vmatrix} = (3\omega_0^2-\omega^2)(\omega_0^2-\omega^2)-\omega_0^4$$

$$C_1 \begin{vmatrix} F_0/M & -\omega_0^2 \\ 0 & \omega_0^2-\omega^2 \end{vmatrix} = \frac{F_0}{M}(\omega_0^2-\omega^2) \qquad C_2 \begin{vmatrix} 3\omega_0^2-\omega^2 & F_0/M \\ -\omega_0^2 & 0 \end{vmatrix} = \omega_0^2\frac{F_0}{M}$$

$(3\omega_0^2-\omega^2)(\omega_0^2-\omega^2)-\omega_0^4 = 3\omega_0^4 - 3\omega_0^2\omega^2 - \omega^2\omega_0^2 + \omega^4 - \omega_0^4 = 0$

$\Rightarrow 2\omega_0^4 - 4\omega_0^2\omega^2 + \omega^4 = 2\omega_0^4 = (-2\omega_0^2+\omega^2)^2$

$(\omega^2 - 2\omega_0^2)^2 = 2\omega_0^4$

$\omega^2 = \pm\sqrt{2}\omega_0^2 + 2\omega_0^2$ define as $\omega_H^2 = (2+\sqrt{2})\omega_0^2$ $\omega_L^2 = (2-\sqrt{2})\omega_0^2$

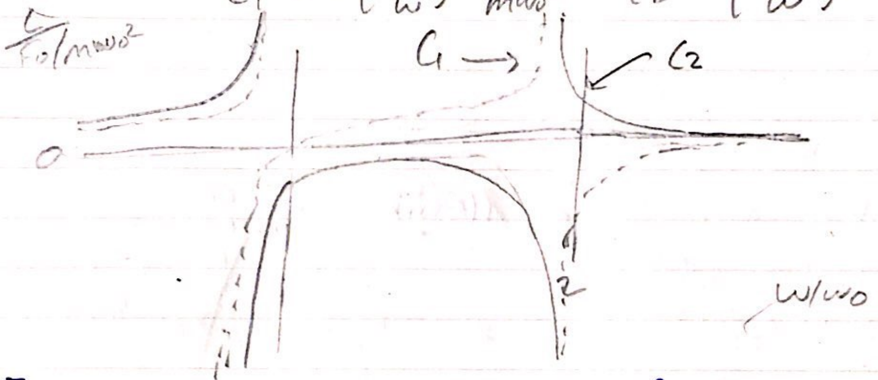
$\Rightarrow D = (\omega^2-\omega_H^2)(\omega^2-\omega_L^2)$

$$C_1 = \frac{F_0/M\,(\omega_0^2-\omega^2)}{(\omega^2-\omega_H^2)(\omega^2-\omega_L^2)} \qquad C_2 = \frac{(F_0/M)\,\omega_0^2}{(\omega^2-\omega_H^2)(\omega^2-\omega_L^2)}$$

If $\omega = 0$ $C_1 = \dfrac{F_0/M\,\cancel{\omega_0^2}}{(-2-\sqrt{2})\omega_0^2\,(-2+\sqrt{2})\cancel{\omega_0^2}} = \dfrac{F_0/M}{(4-2)\omega_0^2} = \dfrac{1}{2}\dfrac{F_0}{M\omega_0^2}$

$C_2 = \dfrac{(F_0/M)\,\cancel{\omega_0^2}}{(-2-\sqrt{2})\omega_0^2(-2+\sqrt{2})\cancel{\omega_0^2}} = \dfrac{F_0/M}{(4-2)\omega_0^2} = \dfrac{1}{2}\dfrac{F_0}{m\omega_0^2}$

as $\omega \to \infty$ $C_1 \to -\left(\dfrac{\omega_0}{\omega}\right)^2 \dfrac{F_0}{m\omega_0^2}$ $C_2 \to \left(\dfrac{\omega_0}{\omega}\right)^4 \dfrac{F_0}{m\omega_0^2}$



d) By inspection of results of c), give the frequencies & amplitude ratios for normal modes of system.

$C_2/C_1 = \dfrac{\omega_0^2}{\omega_0^2-\omega^2}$

for low frequency $\left.\dfrac{C_2}{C_1}\right|_L = \dfrac{\omega_0^2}{\omega_0^2-(2-\sqrt{2})\omega_0^2} = \dfrac{\omega_0^2}{\omega_0^2-2\omega_0^2+\sqrt{2}\omega_0^2} = \dfrac{1}{-1+\sqrt{2}} = 2.414$

high frequency $\left.\dfrac{C_2}{C_1}\right|_H = \dfrac{\omega_0^2}{\omega_0^2-(2+\sqrt{2})\omega_0^2} = \dfrac{\omega_0^2}{\omega_0^2-2\omega_0^2-\sqrt{2}\omega_0^2} = \dfrac{1}{-1-\sqrt{2}} = -0.414$

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