12/26/19 8.03 Pset 3 **Problem 1: Exact behavior near Resonance**

free decay of amplitude of mass-spring oscillator has zeros spaced by 422.6 μs. Same oscillator driven by harmonic force applied to the mass at a freq. that gives max. response shows 3.000 cm at frequency ν = 1,166.2 Hertz.

resonant freq.

$\frac{\text{peak energy in resonator}}{\text{energy lost / radian}}$

A) Find undamped natural frequency ν_0 & Q of oscillator.

$T = \frac{1}{\nu}$ where $\nu = \frac{\omega}{2\pi}$ $\Rightarrow T = \frac{2\pi}{\omega}$ $\omega_1 = \frac{2\pi}{T}$

$\Rightarrow \omega_1 = \frac{2\pi}{422.6\mu s}$ $\omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4} = \left(\frac{2\pi}{422.6\mu s}\right)^2$

$A \propto \frac{1}{\sqrt{\gamma^2\omega_M^2 + (\omega_0^2 - \omega_M^2)^2}}$ $\gamma^2\omega_M^2 + (\omega_0^2 - \omega_M^2)^2 = B$ $\frac{\gamma^2}{2} - \omega_0^2 + \omega_M^2 = 0$

$\frac{d}{d\omega} B^{-1/2} = \cancel{-\frac{1}{2}} B^{-3/2}\left(\cancel{2}\gamma^2\cancel{\omega_M} + 2(\omega_0^2 - \omega_M^2)(-2\cancel{\omega_M})\right) = 0$ $\gamma^2 - 2(\omega_0^2 - \omega_M^2) = 0$

$2\pi(1166.2) = 2\pi(\nu_{max}) = \omega_M^2 = \omega_0^2 - \frac{\gamma^2}{2}$

$\omega_1^2 - \omega_M^2 = \left(\omega_0^2 - \frac{\gamma^2}{4}\right) - \left(\omega_0^2 - \frac{1}{2}\gamma^2\right) = \frac{1}{2}\gamma^2 = \left(\frac{2\pi}{422.6\mu s}\right)^2 - 2\pi(1166.2)$

$= 2.2105 \times 10^8$

$\omega_0^2 = \omega_1^2 + \frac{\gamma^2}{4} = \omega_1^2 + \left(\frac{\gamma^2}{2}\right)/2$

$= 3.3158 \times 10^8$

$\boxed{\nu_0 = \frac{\omega_0}{2\pi} = 2.898 \text{ kHz}}$

$\gamma = 2.1026 \times 10^4$

$\boxed{Q = \frac{\omega_0}{\gamma} = 0.866}$

☆My answer differs from their answer key, which I think is wrong

b) If a force of same magnitude applied at low frequency ($\nu \ll \nu_0$) what will max excursion of mass be? wtf is max excursion of mass

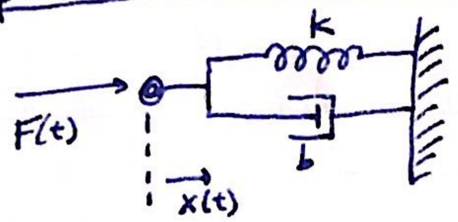
$\Rightarrow$ seems like soln manual asking for amplitude when $\omega = 0$...

$A(0) = \frac{1}{\omega_0^2}$ $A(\omega_M) = \frac{1}{\sqrt{\gamma^2\omega_M^2 + (\omega_0^2 - \omega_M^2)^2}}$ ... follow algebra & use Q for replacement to get A(0)

* This isn't a particularly well-written question. & the solutions full of mistakes. Takeaway:

- "spaced by" means period (T)
- max response frequency can be solved through deriving amplitude & solving for ω_{max}
- $Q = \frac{\omega_0}{\gamma}$

problem 2: Transient Behavior

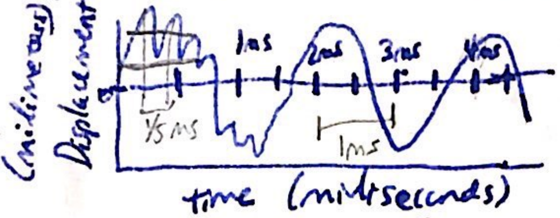


mass $m=1g$ connected to spring & dashpot (source of damping)

driving force:

$$\begin{cases} F(t)=0 & t<0 \leftarrow \text{homogeneous sol'n} \\ F(t)=F_0\cos(\underline{\omega}t+\phi) & t>0 \text{ particular steady state.} \end{cases}$$

subsequent displacement of mass from equilibrium position



use $\pi^2 \approx 10$

aim for accuracy on order of 20%

eq'n of motion: $\ddot{x}+\gamma\dot{x}+\omega_0^2x=\frac{F_0}{m}e^{i\omega t}$

$1N = 1 kg \frac{m}{s^2}$

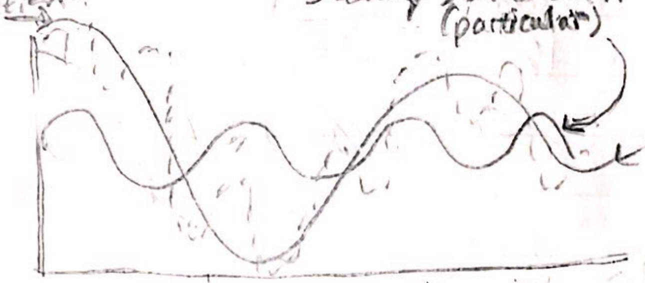
a) Find K in newton-meter^{-1} $\Rightarrow kg/s^2$

Background: $\theta(t)=A(\omega_d)\cos[\omega_d t-\delta(\omega_d)]+Be^{-\gamma/2 t}\cos(\omega t+\alpha)$

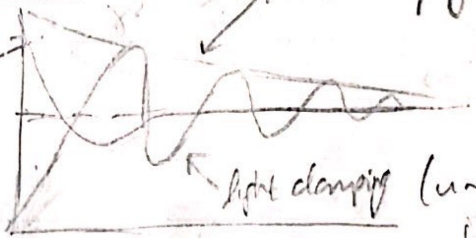
↑ steady state sol'n (particular)

↑ homogeneous sol'n (set eq'n of motion $=0$)

Superposition



↳ shows light damping:



hard damping (overdamping) $\frac{\gamma^2}{4}>\omega_0^2$

Critically damped $\frac{\gamma^2}{4}=\omega_0^2$

light damping (underdamping) $\omega_0^2>\frac{\gamma^2}{4}$

In this case, because the homogeneous sol'n is decreasing!  → underdamped where $\omega_0^2>\frac{\gamma^2}{4}$.

We make approximation that $\omega_1 \approx \omega_0$ (where $\omega_1=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}$)

$\omega_1=\frac{2\pi}{T}=\frac{2\pi}{1/5ms}=\frac{10\pi}{ms}\times\frac{1000ms}{1s}=10^4\pi s^{-1}$

$\omega_0=\sqrt{\frac{k}{m}} \Rightarrow k=\omega_0^2m=[10^4\pi s^{-1}]^2[0.001 kg]=10^8(10)(0.001)\frac{kg}{s^2}$

$=10^9 10^{-3}=\boxed{10^6\frac{kg}{s^2}}$

b) Find b in newton-second-meter^{-1}

$\frac{kg\,m}{s^2}\,s\,\frac{1}{m}=\frac{kg}{s}$

$\ddot{x}+b\dot{x}+\omega_0^2x=\frac{F}{m}e^{i\omega t}$

$\Rightarrow m\ddot{x}+m\gamma\dot{x}+\omega_0^2m=F_0e^{i\omega t}$

Amplitude of fast oscillation decreases to $\frac{1}{2}$ in 0.6ms

$\frac{1}{\gamma/2}\sim0.6ms$ $\gamma=\frac{2}{0.6\times10^{-3}}\sim3\times10^3$

$=b/m$

$b=m\times3\times10^3=3$ newton-second-m^{-1}

$1\ Hz = 1000\ ms$

c) Find $\frac{\omega}{2\pi}$ in Hertz

From slow oscillations $T = \frac{2\pi}{\omega}$ $T = \frac{1}{\nu}$ $\nu = \frac{1}{T} = \frac{1}{2ms}\ \frac{1000\ ms}{1s}$

$\nu = \frac{\omega}{2\pi}$ $\omega = \frac{2\pi}{T} = 2\pi f$

$= 500\ s^{-1}$

$= 500\ Hz$

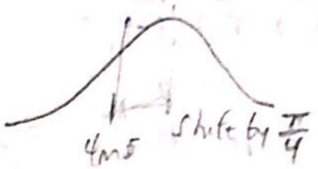
ordinary freq. (Hertz) often symbolized as ν

* Rember ν gives you Hertz, which is just s^{-1}

d) Find ϕ in degrees

Slow periodic w/ $T = 2ms$.

$\phi = -\frac{\pi}{4} = -45°$



cos

$\cos(\omega t - \frac{\pi}{4})$

(i.e. $\cos(0) = 1$

now needs ω to be $\frac{\pi}{4}$

e) Find F_0 in newtons $\frac{kg\ m}{s^2}$

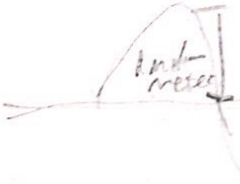
after transient has died out, $\omega \ll \omega_0$

$F = -kx = -10^6 \frac{kg}{s^2}\ 10^{-3} m$

$F_0 = 10^3 N$

"transient has died out refers to the particular solution"

not sure where this came from but is 1 millimeter



spring position...? equilibrium is in the middle

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when $\omega = \omega_0$?: $V_0 = \frac{\omega_0 I/c}{\sqrt{(\omega\gamma)^2}} = \frac{\cancel{\omega_0} I/c}{\cancel{\omega_0}\gamma} = \frac{I_0/c}{\gamma}$

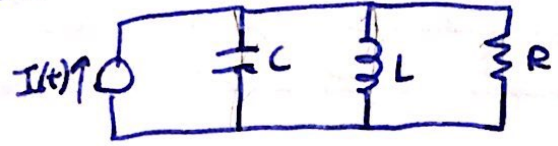
when $\omega = 0$: $V_0 = 0$

when $\omega \to \infty$ $\lim_{\omega \to \infty} V_0 = \frac{\omega\ I/c}{\sqrt{(\gamma\omega)^2 + (\omega_0^2 - \omega^2)^2}} = \frac{I_0/c}{\omega}$

(from problem 4a)

problem 4: Parallel RLC Circuit

12/27/19



object of problem: show that in a harmonically driven system, the frequency of the response depends on which system variable one looks at.

Current source: $I(t) = I_0 \cos \omega t$

Same V across everything

assume Q~4 sketch function behavior at $\omega=0$ $\omega=\omega_0$ & $\omega \to \infty$

a) Voltage across resistor: $V_0 \cos(\omega t - \delta_V)$. Find frequency dependence of amplitude $V_0(\omega)$ & phase $\delta_V(\omega)$ of the voltage?

$I_L = \frac{1}{L}\int V dt$ $I_R = \frac{V}{R}$ $I_L + I_R + I_c = I_0 \cos(\omega t)$ get general sol'n for I, then I×R

$I_C = C\dot{V}$

$= \frac{1}{L}V + \frac{1}{R}\dot{V} + C\ddot{V} = I_0 \frac{d}{dt}(\cos(\omega t))$

$= \ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = \frac{I_0}{C}\frac{d}{dt}(\cos(\omega t))$

$i = -\frac{dQ}{dt} = -\frac{dV}{dt}$

$\cos(\omega t) = Re(e^{i\omega t})$

$P(i\omega) = D^2 + \frac{1}{RC}D + \frac{1}{LC}$

$= -\omega^2 + \gamma i\omega + \omega_0^2$

→ guess sol'n $V_0 e^{i(\omega t - \delta)}$

$v(t) = Re[v]$

$\Rightarrow (\omega_0^2 - \omega^2 + \gamma i\omega) A \frac{e^{i(\omega t - \delta)}}{e^{i(\omega t - \delta)}}$

$= i\omega \frac{I_0}{C} \frac{e^{i\omega t}}{e^{i(\omega t - \delta)}}$

$\Rightarrow (\omega_0^2 - \omega^2 + \gamma i\omega) V_0 = i\omega \frac{I_0}{C} e^{i\delta}$

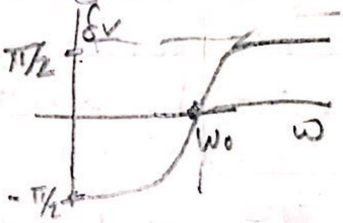
$ie^{i\delta} = (\cos(\delta) + i\sin(\delta));$

$= i\cos(\delta) - \sin(\delta)$

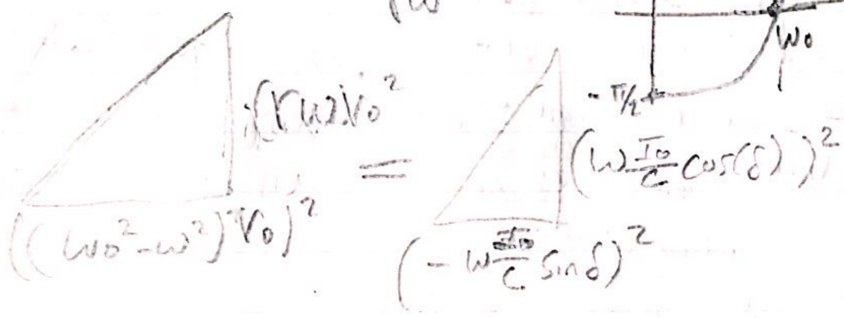
real: $(\omega_0^2 - \omega^2) V_0 = -\omega \frac{I_0}{C} \sin(\delta)$

Im: $\gamma \omega V_0 = \omega \frac{I_0}{C} \cos(\delta)$

$\tan \delta = \frac{\omega^2 - \omega_0^2}{\gamma \omega}$



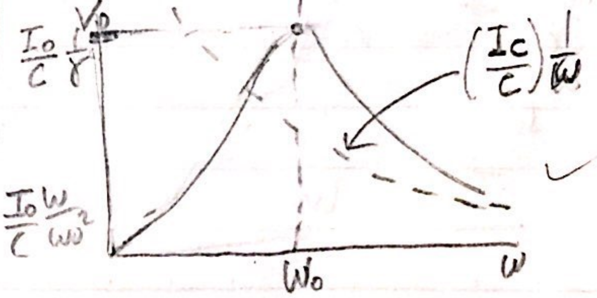
$= \omega \frac{I_0}{C}(i\cos(\delta) - \sin\delta)$



$V_0 \sqrt{[(\gamma\omega)^2 + (\omega_0^2 - \omega^2)]^2} = \sqrt{(\omega \frac{I_0}{C}\cos(\delta))^2 + (-\omega \frac{I_0}{C}\sin(\delta))^2} = \sqrt{(\omega \frac{I_0}{C})^2 (1)}$

$$\Rightarrow \boxed{V_0 = \frac{\omega I_0 / C}{\sqrt{(\gamma\omega)^2 + (\omega_0^2 - \omega^2)^2}}}$$

when $\omega_0 = \omega$, $V_0 = \frac{\omega_0 I/C}{\sqrt{(\gamma\omega_0)^2}} = \frac{\cancel{\omega_0} I/C}{\gamma \cancel{\omega_0}}$



b) The current through the resister has form $\boxed{I_R \cos(\omega t - \delta_R)}$ Find frequency dependence of the amplitude $I_R(\omega)$ and phase $\delta_R(\omega)$. Sketch results.

$$I_R \cos(\omega t - \delta_R) = = \frac{V_0 \cos(\omega t - \delta_v)}{R}$$

$$\boxed{I_R = \frac{V_0}{R} \qquad \delta_R = \delta_V}$$

$$I_R \cos(\omega t - \delta_R) \ldots = \frac{V_0}{R} \cos(\omega t - \delta_R)$$

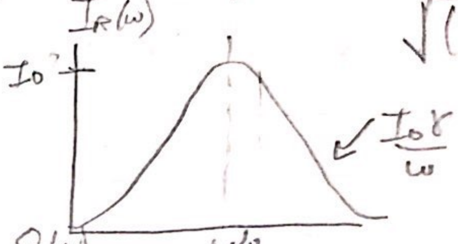
$\omega \to \omega_0$

$$\frac{V_0}{R} = \frac{\omega_0 I/c}{\gamma \omega_0} = \frac{I}{C\gamma} = \textcircled{I_0}$$

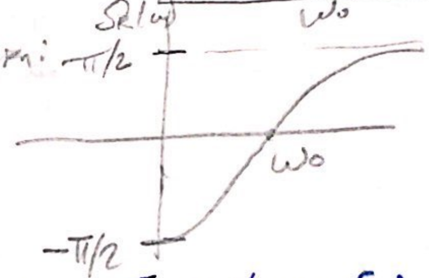
not sure why defined as I_0... $V_0 = \dfrac{\omega I/c}{\sqrt{(\gamma\omega)^2 + (\omega_0^2 - \omega^2)^2}}$

$\omega \to \infty$

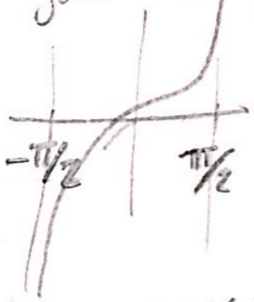
$$\frac{V_0}{R} = \frac{I}{C\omega R} = \frac{I_0 \gamma}{\omega}$$



Normal tan function:



(normal tan graph goes from $-\frac{\pi}{2} \to \frac{\pi}{2}$)



c) Current through capacitor: $I_c \cos(\omega t - \delta_c)$ sketch $I_c(\omega)$ and $\delta_c(\omega)$

$$C\dot{V} = I_c = -C V_0 \omega \sin(\omega t - \delta_v)$$
$$= C V_0 \omega \cos(\omega t - \delta_v + \pi/2)$$
$$= I_c \cos(\omega t - \delta_c)$$

$\underline{I_c = C V_0 \omega}$ $\quad$ $\underline{\delta_c = \delta_v - \pi/2}$

$\cos(x) = \sin(x + \frac{\pi}{2})$

$\sin(x) = -\cos(x + \frac{\pi}{2})$

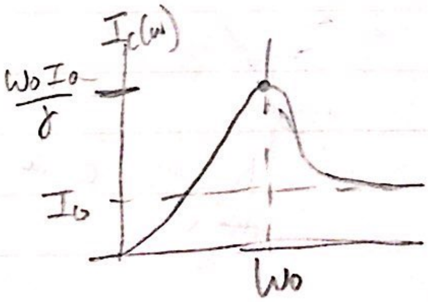
$= \cos(x - \frac{\pi}{2})$

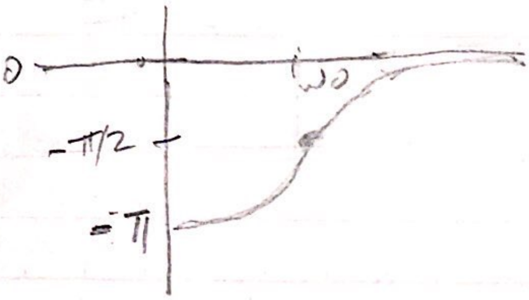
$$I_c = \frac{\omega^2 I}{\sqrt{(\gamma\omega)^2 + (\omega_0^2 - \omega^2)^2}}$$

when $\omega \to \omega_0$

$$\frac{\omega_0^2 I}{\gamma \omega_0} = \frac{\omega_0 I_0}{\gamma}$$

when $\omega \to \infty \Rightarrow I_0$





d) Current through inductor: $I_L \cos(\omega t - \delta_L)$. Find $I_L(\omega)$ & $\delta_L(\omega)$, sketch results

$$I_L = \frac{1}{L}\int V dt = \frac{1}{L}\int V_0 \cos(\omega t - \delta_V)\, dt = I_L \cos(\omega t - \delta_L)$$

$$= \frac{V_0}{L\,\omega} \sin(\omega t - \delta_V) = I_L \cos(\omega t - \delta_L)$$

$$= \frac{V_0}{\omega L} \cos(\omega t - \delta_V - \tfrac{\pi}{2})$$

moral: always set I_L etc. to be positive because this is equivalent to $\frac{-V_0}{\omega L}\cos(\omega t - \delta_V + \frac{\pi}{2})$

$$\Rightarrow I_L = \frac{V_0}{\omega L} \quad \delta_L = \delta_V + \frac{\pi}{2}$$

$$I_L = \frac{I/LC}{\sqrt{(\gamma\omega)^2 + (\omega_0^2 - \omega^2)^2}}$$

$\omega \to \omega_0$

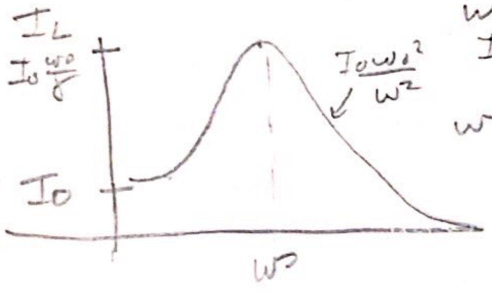
$$I_L = \frac{I/LC}{\gamma\omega_0} = \frac{I_0\,\omega_0}{\gamma}$$

$\omega \to 0$

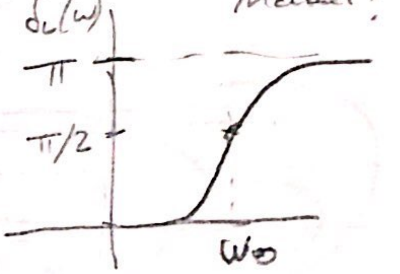
$$I_L = I_0 = \frac{I/LC}{\omega_0^2}$$

$\omega \to \infty$

$$I_L = \frac{I/LC}{\omega^2} = \frac{I_0\,\omega_0^2}{\omega^2}$$



How is I_0 defined? does it just not matter?



e) - inductor behaves like short circuit at low frequencies, open circuit at high freq.

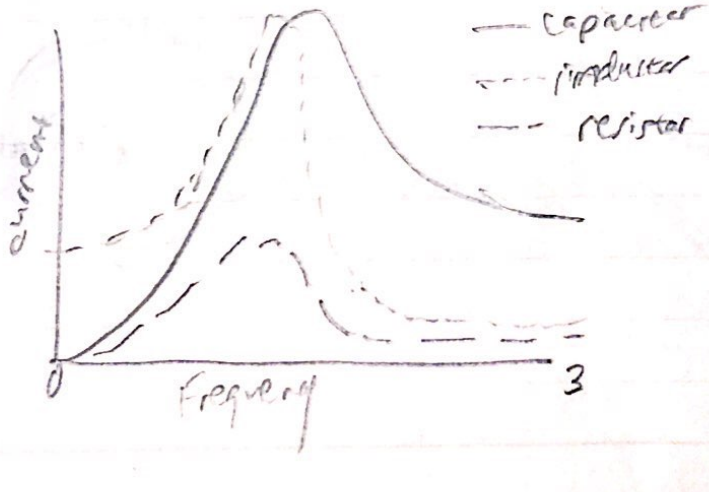
- capacitor behaves like open circuit at low freq. & short circuit at high freq. How are your results for current amplitude & phase for inductor & capacitor consistent with this view?

→ inductor at low ω → $= I_0$ all current goes through it & none through C or R.

"no phase lags at $\omega = 0$

capacitor at high ... same ...

at $\omega = \omega_0$, large stored current $(\frac{\omega_0}{\gamma}) I_0$ sloshes between L & C, bypassing R, which only carries I_0



Current in parallel RLC circuit with Q=3

$$\left(Q = \frac{\omega_0}{\gamma}\right)$$

Pset 3 Problem 5 Transient Behavior of a Critically Damped System 12/28/19

Critically damped mass-~~or~~ spring oscillator initially at rest is set into vibration by driving the mass w/ harmonic force at radian frequency ω.

a) Find exact expression for $x(t)$ in terms of steady state amplitude A, phase shift δ, & undamped natural freq. ω_0.

general: $(A+Bt)e^{-\frac{\gamma}{2}t}$ particular sol'n $A(\omega_d)\cos[\omega_d t - \delta(\omega_d)]$

$$\boxed{(C_1+C_2t)e^{-\frac{\gamma}{2}t} + A\cos(\omega t-\delta)}$$

b) Simplify expression for when $\omega=\omega_0$. Sketch how first cycle of response modified by transient behavior [Hint: what is δ when $\omega=\omega_0$?]

Fill in C_1, C_2 & A $x(0)=\dot{x}(0)=0$

$C_1 = -A\cos(-\delta) = A\cos(\delta)$ $\dot{x}(0)=0=\left[C_1e^{-\frac{\gamma}{2}t} + C_2te^{-\frac{\gamma}{2}t} + A\cos(\omega t-\delta)\right]'$

$= C_1\left(-\frac{\gamma}{2}\right)e^{-\frac{\gamma}{2}t} + C_2e^{-\frac{\gamma}{2}t} + C_2t\left(-\frac{\gamma}{2}\right)e^{-\frac{\gamma}{2}t}$

$- A\omega\sin(\omega t-\delta)$

$= -C_1\frac{\gamma}{2} + C_2 + A\omega\sin(\delta)$

$\Rightarrow C_2 = C_1\frac{\gamma}{2} - A\omega\sin(\delta)$

$\left[-A\cos(\delta) + \left(C_1\frac{\gamma}{2} - A\omega\sin(\delta)\right)t\right]e^{-\frac{\gamma}{2}t}$

$+ A\cos(\omega t-\delta)$

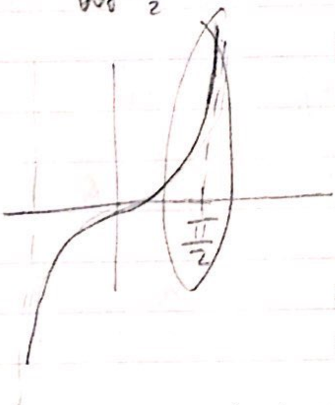
$$\boxed{\begin{aligned}x(t) &= -A\left[\cos(\delta) + (\omega_0\cos(\delta) \\ &\quad + \omega\sin(\delta))t\right]e^{-\omega_0 t} \\ &\quad + A\cos(\omega t-\delta)\end{aligned}}$$

In critically damped, $\omega_0^2 = \left(\frac{\gamma}{2}\right)^2$

$\omega_0 = \frac{\gamma}{2}$

$\tan\delta_{\omega\to\omega_0} = \frac{\gamma\omega}{\omega_0^2-\omega^2} = \frac{\gamma\omega_0^2}{0}$

$\delta = \frac{\pi}{2}$ at $\omega=\omega_0$



$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

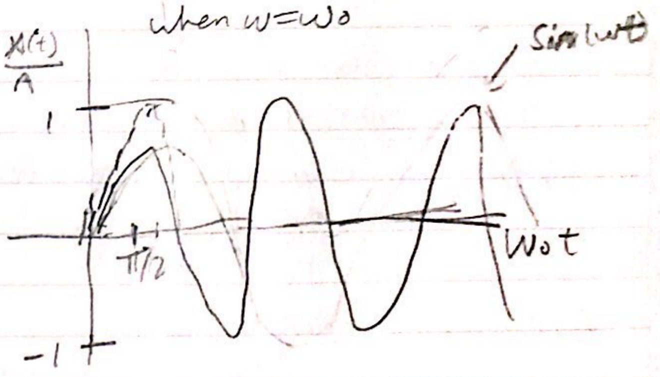
$x(t) = -A[\omega t]e^{-\omega_0 t}$

$+ A\cos\left(\omega t - \frac{\pi}{2}\right)$

$= -A\omega te^{-\omega t} + A\sin(\omega t)$

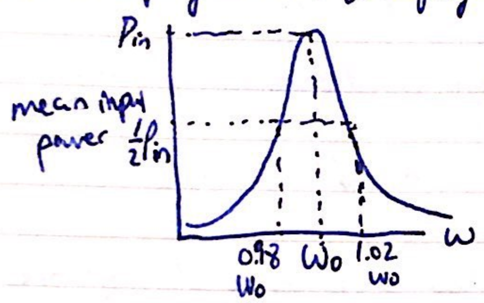
$= A(\sin(\omega t) - \omega te^{-\omega t})$

when $\omega=\omega_0$



Problem 6

Figure shows mean power input $\bar{P}$ as a function of driving frequency for a mass on a spring with damping:



constant

varied

Driving force = $F_0 \sin \omega t$

Q high so mean power input, which is max at ω_0, falls to half-maximum at $0.98\,\omega_0$ & $1.02\,\omega_0$

a) What is the numerical value of Q?

$$Q = \frac{\omega_0}{\gamma}$$

Background

$$P = Fv \qquad F = F_0 \cos(\omega t) \qquad v = \frac{dx}{dt} = v_0 \sin(\omega t - \delta)$$

$$P = F_0 v_0 \cos(\omega t) \sin(\omega t - \delta) = F_0 v_0 [\cos^2(\omega t) \sin(\delta) - \sin(\omega t)\cos(\omega t)\cos(\delta)]$$

P is instantaneous power, We want averages:

$$\langle \cos^2(\omega t) \rangle = \frac{1}{2\pi} \int_{\omega t = 0}^{2\pi} \cos^2(\omega t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} (1 + \cos(2\omega t))\, dt = \frac{1}{2}$$

$$\langle \sin(\omega t) \cos(\omega t) \rangle = 0$$

$$\langle P \rangle = \frac{1}{2} F_0 v_0 \sin\delta \qquad |v| = \omega A \qquad \sin\delta = \frac{\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}}$$

$$\langle P \rangle = \frac{1}{2} F_0 \frac{m^{-1} \omega F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \frac{\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}}$$

$$= \frac{F_0^2}{2m} \frac{\gamma\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

We can maximize power transferred into a system by driving at resonance.

$$= \frac{F_0^2}{2m\omega_0^2} \left[\frac{\gamma}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}} \right]$$

the denominator is maximized at resonance when $\omega = \omega_0$ in which case the avg power is:

$$\langle P \rangle_{max} = \frac{F_0^2 Q}{2m\omega_0} \propto Q$$

How do we excite a single mode (one ω_0 peak) system? By driving close to ω_0. How close to ω_0 is close?

$\frac{1}{2}\langle P \rangle_{max}$ as threshold for excitement. At half-maximum:

$$\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} = \frac{1}{Q}$$

Approximating $\omega \approx \omega_0$, we have:

$$\frac{\omega_0^2 - \omega^2}{\omega_0 \omega} \approx \frac{(\omega_0 + \omega)(\omega_0 - \omega)}{\omega_0^2} \approx \frac{2\omega_0 \Delta\omega_{half}}{\omega_0^2} = \frac{2\Delta\omega_{half}}{\omega_0}$$

half width is $\Delta\omega_{half} = \frac{\omega_0}{2Q}$ so full width at half-maximum is

$$\Delta\omega_{FWHM} \approx \frac{\omega_0}{Q}$$

(approximate) $$\bar{P}(\omega) = \frac{\gamma F_0^2}{2m} \frac{1}{4(\omega_0 - \omega)^2 + \gamma^2}$$

avg power of driven oscillator:

$$\vec{P}(\omega) = \frac{\gamma F_0^2}{2m} \frac{1}{4(\omega_0-\omega)^2+\gamma^2} \quad \text{at resonance} \rightarrow \vec{P}_{max} = \frac{F_0^2}{2m\gamma}$$

at half maximum

$$2 = \frac{\vec{P}_{max}}{\vec{P}(\omega)} = \frac{\cancel{F_0^2}/\cancel{2m}\gamma}{\gamma \cancel{F_0^2}/\cancel{2m} \quad 1/4(\omega_0-\omega)^2+\gamma^2}$$

$$\gamma^2 = 2\gamma^2 - \gamma^2 = 4(\omega_0-\omega)^2 \Longleftarrow 2 = \frac{4(\omega_0-\omega)^2+\gamma^2}{\gamma^2}$$

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{2(\omega_0-\omega)} = \frac{\omega_0}{0.04\omega_0} = \boxed{25}$$

b) If driving force is removed, the energy decreases according to $\boxed{E = E_0 e^{-\gamma t}}$. What is the value of γ?

$\gamma = 0.04\omega_0$

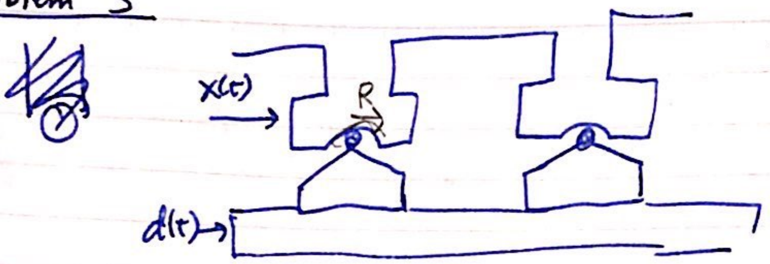
c) If driving force is removed, what fraction of energy is lost /cycle?

relative energy loss:

$$\frac{1}{E}\frac{dE}{dt} = \frac{1}{E}\left(-\gamma E_0 e^{-\gamma t}\right) = -\gamma \quad \underline{\text{integrate over one period } \Delta t = \frac{2\pi}{\omega_0}}$$

$$\frac{1}{E}\frac{\Delta E}{\Delta t} = -\gamma \Rightarrow \frac{\Delta E}{E} = -\gamma \Delta t = -\gamma \frac{2\pi}{\omega_0} = -\frac{0.04\cancel{\omega_0}\,2\pi}{\cancel{\omega_0}} = \boxed{-0.08\pi}$$

Problem 3



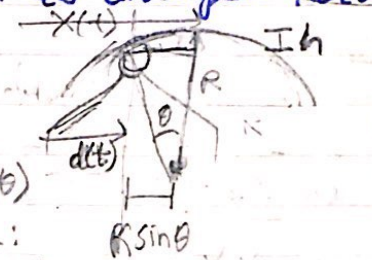
Friction between foundation block & column base

$F_x(\text{friction}) = -b\frac{d}{dt}(x(t) - d(t))$

$F_x = -b\frac{d}{dt}x(t) + b\frac{d}{dt}d(t)$

a) At equilibrium, tip of foundation block is at center of spherical depression. If building is displaced horizontally, its center of gravity will rise. Write down differential eqn relating horizontal pos. of building in inertial frame x(t) to position of earth in frame d(t).
Give expression for undamped natural frequency of building.

building columns pendulums



$\uparrow F_{restoring}$

$F_x \rightarrow$

friction causes some amount of x(t) to move w/ d(t)

$h = R - R\cos\theta = R(1-\cos\theta)$

small angle approximation:

$h = R(1 - [1 - \frac{\theta^2}{2}]) = R(\frac{\theta^2}{2})$

$\sin\theta \approx \theta \approx \frac{x-d}{R}$

$\rightarrow h = R\frac{(x-d)^2}{R^2}/2 \Rightarrow \frac{(x-d)^2}{2R}$

$V = mg\Delta h$
$= mg\frac{(x-d)^2}{2R}$

m = mass of the column
⇒ f fraction of total building weight.

$F_{restoring} = -\frac{dV}{dx} = -\frac{mg\,2(x-d)}{\cancel{2}R} = -\frac{mg}{R}(x-d)$

$F_x(\text{friction}) = -b\frac{d}{dt}x(t) + b\frac{d}{dt}d(t)$

} $\sum F = F_{restoring} + F_x(\text{friction})$

$\sum F = ma = m\ddot{x} \Rightarrow m\ddot{x} = -\frac{mg}{R}(x-d) - b\dot{x} + b\dot{d}$

* ω for pendulum does not depend on the mass

$\Rightarrow m\ddot{x} + b\dot{x} + \frac{mg}{R}x = b\dot{d} + \frac{mg}{R}d$

$\Rightarrow \boxed{\ddot{x} + \frac{b}{m}\dot{x} + \frac{g}{R}x = \frac{b}{m}\dot{d} + \frac{g}{R}d}$

$\underbrace{\frac{g}{R}}_{=\omega_0^2} \Rightarrow \boxed{\omega_0 = \sqrt{\frac{g}{R}}}$

b) If the ~~building~~ earth shakes $d(t) = d_0\cos(\omega t)$, the building will shake at the same frequency $x(t) = X_0(\omega)\cos(\omega t - \delta(\omega))$ particular sol'n
Find complex algebraic eq'n for $X_0(\omega)$ and $\delta(\omega)$ which follows from differential eq'n for pt. A.

$d(t) = d_0\cos(\omega t) = d_0 \operatorname{Re}[e^{i\omega t}]$

$X(t) = X_0(\omega)\cos(\omega t - \delta(\omega)) = X_0(\omega)\operatorname{Re}[e^{i(\omega t - \delta(\omega))}]$

$\ddot{x} + \frac{b}{m}\dot{x} + \frac{g}{R}x = \frac{b}{m}\dot{d} + \frac{g}{R}d \Rightarrow [\omega_0^2 - \omega^2 + \gamma i\omega]X_0(\omega)\operatorname{Re}[e^{i(\omega t - \delta(\omega))}]$

$= [\gamma i\omega + \omega_0^2]\, d_0 \operatorname{Re}[e^{i\omega t}]$

$\boxed{\Rightarrow [\omega_0^2 - \omega^2 + \gamma i\omega]X_0 = [\omega_0^2 + \gamma i\omega]d_0 e^{i\delta}}$

c) Find expression for transmission function $X_0(\omega)/d_0$. Plot result. Does it have expected behavior at ω=0? What is the limiting form as ω→∞? $e^{i\delta} = \cos\delta + i\sin\delta$ cancels out

$\frac{X_0}{d_0} = \frac{[\omega_0^2 + \gamma i\omega]}{[\omega_0^2 - \omega^2 + \gamma i\omega]}\cancel{e^{i\delta}} = \frac{\sqrt{\omega_0^4 + \gamma^2\omega^2}}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$

$\frac{X_0(0)}{d_0} = \frac{\sqrt{\omega_0^4}}{\sqrt{\omega_0^4}} = 1$

$$\lim_{\omega\to\infty}\frac{X_0(\omega)}{d_0} = \frac{\sqrt{\gamma^2\omega^2}}{\sqrt{\omega^4+\gamma^2\omega^2}} = \frac{\sqrt{\gamma^2}}{\sqrt{\omega^2+\gamma^2}} = \frac{\gamma}{\omega}$$ apparently ? Don't understand why.

d) Show the transmission function is unity at $\boxed{\omega=\sqrt{2}\,\omega_0}$ independent of the value of b. How does the transmission at $\underline{\omega=\omega_0}$ depend on the Q of the system? $Q=\frac{\omega_0}{\gamma}$

$$\left[\frac{\omega_0^4+(\gamma\omega)^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right]^{1/2} = \left[\frac{1+\frac{\gamma^2\omega^2}{\omega_0^4}}{\underbrace{\frac{(\omega_0^2-\omega^2)^2}{\omega_0^4}}+\frac{\gamma^2\omega^2}{\omega_0^4}}\right]^{1/2}$$

$$\frac{\gamma^2\omega^2}{\omega_0^4} = \frac{\omega^2}{\omega_0^2\,Q^2}$$

$$= \left(\frac{\omega_0^2-\omega^2}{\omega_0^2}\right)^2 = \left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2$$

$$= \left[\frac{1+\left(\frac{\omega}{\omega_0}\right)^2/Q^2}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(\frac{\omega}{\omega_0}\right)^2/Q^2}\right]^{1/2}$$

$$\omega=\sqrt{2}\,\omega_0 \Rightarrow \left[\frac{1+\overbrace{\left(\frac{\sqrt{2}\,\omega_0}{\omega_0}\right)^2}^{2}/Q^2}{\left(1-\underbrace{\left(\frac{\sqrt{2}\,\omega_0}{\omega_0}\right)^2}_{2}\right)^2+\underbrace{\left(\frac{\sqrt{2}\,\omega_0}{\omega_0}\right)^2}_{2}/Q^2}\right]^{1/2}$$

$$\boxed{\frac{X_0(\omega=\sqrt{2}\,\omega_0)}{d_0}=1}$$

$$\boxed{X_0(\omega=\omega_0)/d = \left[\frac{1+1/Q^2}{0+1/Q^2}\right]^{1/2} = \sqrt{Q^2+1}}$$