8.03 Pset 2

problem 1

Key concept: These are damped harmonic oscillators $\Rightarrow$ damped slows down motion

12/23/19

displacement $s(t)$ from equilibrium of pen ... obeys eq'n for damped harmonic oscillators

$$\ddot{s}(t) + \gamma\dot{s}(t) + \omega_0^2 s(t) = 0$$

a) Find ~~find z(t)~~ time evolution of displacement if pen is critically damped & $s(0)=0$ & $\dot{s}(0)=V_0$. Does s change sign before equilibrium position at $s=0$

$$\frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \Rightarrow \left.\right] \Rightarrow e^{-\frac{\gamma}{2}t} =$$

set sqrt $=0$ $\quad \frac{\gamma^2}{4} = \omega_0^2 \Rightarrow \omega_0 = \frac{\gamma}{2} \quad \gamma = 2\omega_0$

$$z(t) = Ae^{-\frac{\gamma}{2}t}$$

$$H^2 + \gamma H + \omega_0^2 = 0 = H^2 + 2\omega_0 + \omega_0^2 = 0 = (H+\omega_0)^2$$

$$(\ddot{s} + 2\omega_0\dot{s} + \omega_0^2 s)e^{\omega_0 t} = 0$$

$$= \frac{d^2}{dt^2}(Xe^{\omega_0 t}) = \left(\frac{dx}{dt}e^{\omega_0 t} + X\omega_0 e^{\omega_0 t}\right) = \frac{dx^2}{dt}(e^{\omega_0 t}) + \frac{dx}{dt}(\omega_0 e^{\omega_0 t}) + \omega_0\frac{dx}{dt}e^{\omega_0 t} + X\omega_0^2 e^{\omega_0 t} = \frac{dx^2}{dt^2} + (2\omega_0 e^{\omega_0 t})\frac{dx}{dt} + \omega_0^2 x$$

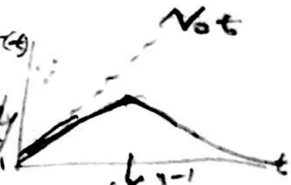
learned in 18.03 where duplicate roots: $e^{-t}, te^{-t}, t^2e^{-t}$ etc.

$$Xe^{\omega_0 t} = A + Bt \Rightarrow X = (A+Bt)e^{-\omega_0 t}$$

$$s(0) = 0 = (A + Bt)e^0 \Rightarrow A = 0$$

$$\dot{s}(0) = V_0 = Be^{-\omega_0 t} + \cancel{(A+Bt) - \omega_0 e^{-\omega_0 t}} \Rightarrow B = V_0$$

$$s = V_0 t e^{-\omega_0 t} = \boxed{V_0 t e^{-\frac{\gamma}{2}t}}$$



Note $s(t) > 0 \ \forall t$ $\Rightarrow$ no overshoot returning to $s=0$

No change sign before settling to $s=0$

approximately a Gaussian?

$\int e^{-x^2}dx \Rightarrow$ Gaussian $\left(\frac{\gamma}{2}\right)^{-1}$

b) Find the response of an overdamped pen to initial conditions $s(0)=s_0$ & $\dot{s}(0)=0$

overdamped: $\frac{\gamma^2}{4} - \omega_0^2 > 0$

Sol'ns: $z(t) = Ae^{-T_1 t} + Be^{-T_2 t}$ where $T_{1,2} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

$z(0) = A + B = s_0$

$\dot{z}(t) = -T_1 A e^{-T_1 t} - T_2 B e^{-T_2 t}$

$\dot{z}(0) = 0 = -T_1 A - T_2 B = 0 \qquad -T_1 A = T_2 B$

$B = -\frac{T_1}{T_2}A$

$A - \frac{T_1}{T_2}A = s_0$

$\Rightarrow A = \frac{s_0}{1 - \frac{T_1}{T_2}} \qquad B = -\frac{T_1}{T_2}\frac{s_0}{1 - \frac{T_1}{T_2}}$

$$\boxed{z(t) = \frac{s_0}{1 - \frac{T_1}{T_2}}\left(e^{-T_1 t} - \frac{T_1}{T_2}e^{-T_2 t}\right)}$$

c) For ~~what~~ does s overshoot its equilibrium position at $s=0$?

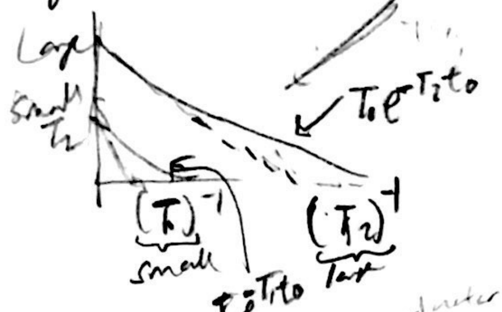
if $s(t)=0$, $\quad A\left(e^{-T_1 t} - \frac{T_1}{T_2}e^{-T_2 t}\right) \Rightarrow \frac{T_1}{T_2}e^{-T_2 t} = e^{-T_1 t}$

$$T_1 e^{-T_2 t} = T_2 e^{-T_1 t}$$

Scanned with CamScanner

assume $T_1 > T_2$

dead cross... no overshoot.



Problem 2 $\xrightarrow{i(t)}$ Simple Electrical Circuit

$C = 1\ \mu F \quad L = 1\ mH \quad R = 10\ \Omega$ If $V_c(0) = 10V = V_0$ & $i(0) = 500\ mA$

Vcapacitor

Find $V_c(t)$ for $t>0$

For each element
$V_L = L\frac{di}{dt}$
$V_R = -iR$
$V_c = \frac{1}{C}\int i\,dt$

$V_L + V_c + V_R = 0 \Rightarrow L\frac{di}{dt} + \frac{1}{C}\int i\,dt + iR = 0 \quad i = -\frac{dQ}{dt} = C\frac{dV}{dt}$

$\underbrace{\quad}_{CV = Q}$ current flowing opposite to capacitor

$LC\frac{d^2V_c}{dt^2} + \frac{1}{C}(CV_c) + C\frac{dV_c}{dt}R = 0$

$\Rightarrow LC\frac{d^2V_c}{dt^2} + V_c + C\frac{dV_c}{dt}R = 0 = e^{0t}$

$+V_L - V_R - V_c = 0$

$\left(L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt\right) = 0$

$\ddot{V}_c + \frac{R}{L}\dot{V}_c + \frac{1}{LC}V_c = 0$ here we always want + signs

$\omega_0 = \frac{1}{\sqrt{LC}}$

$D^2 + \underbrace{\frac{R}{L}}_{\gamma}D + \underbrace{\frac{1}{LC}}_{\omega_0^2} \Rightarrow$ roots $= \dfrac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$

$s^2 + \frac{R}{L}s + \frac{1}{LC} = e$

$= \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{L}\right)^2/4 - \omega_0^2}$

$i = -\frac{dQ}{dt} = -C\frac{dV_c}{dt}$

roots $= -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \frac{1}{LC}} \Rightarrow e^{-\frac{\gamma}{2}t}e^{\pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t}$

$\Rightarrow e^{-\frac{\gamma}{2}t}\left(C_1\cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\right) + C_2\sin\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\right)\right)$

conventions
resonance freq.
$\omega_0 = \frac{1}{LC}$
damped resonant freq.
$\omega_d = \sqrt{\omega_0 - \frac{\gamma^2}{4}}$

$V_c(t) = Ae^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)$ where $\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

$V_c(0) = V_0\cos(\phi) = 10 \quad \Rightarrow$ call A V_0

$\omega_0 = \frac{1}{\sqrt{LC}} = (10^{-3} \times 10^{-6})^{\frac{1}{2}} = \sqrt{10} \times 10^4$

$i = -C\frac{dV}{dt} = -CV_0\left(e^{-\frac{\gamma t}{2}}\cos(\omega_d t + \phi)\right)'$

$\gamma = \frac{R}{L} = \frac{10}{10^{-3}} = 10^4$

$i(t) = CV_0\left(-\frac{\gamma}{2}e^{-\frac{\gamma t}{2}}\cos(\omega_d t + \phi) + e^{-\frac{\gamma t}{2}}\omega_d\sin(\omega_d t + \phi)\right)$

$i(0) = CV_0\left(\frac{\gamma}{2}\cos\phi + \omega_d\sin(\phi)\right) = 0.5 \quad \Rightarrow \underbrace{\frac{C\gamma}{2}}_{\frac{10^{-2}}{2}}\underbrace{V_0\cos\phi}_{10} + (C\omega_d)\underbrace{V_0}_{=\frac{10}{\cos\phi}}\sin\phi = 0.5$

$0.05 + C\omega_d\, 10\tan\phi = 0.5$
solve for ϕ.

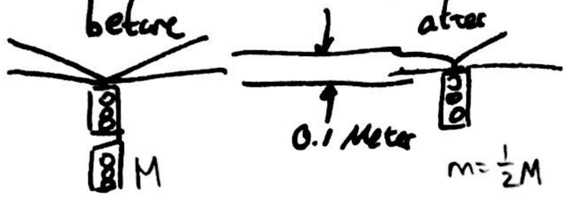
8.03 sol'n manual is wrong fyi!

Now $V_0 = \frac{10}{\cos\phi}$ full eq'n $\boxed{V_c(t) = V_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)}$ this is ω_d not ω_0

$\omega_d = \sqrt{\omega_0 - \frac{\gamma^2}{4}}$

Pset 2 12/24/19

Question 3: Oscillating Traffic Light



traffic signal oscillates.
damped mass-spring oscillator
wire → spring
amplitude disturbances took 4 seconds to damp out
lower half of signal broke. New equilibrium position

a) Undamped natural freq. (in Hertz) after separation.

$y=0$, 0.1 m

$\ddot{x}+\gamma\dot{x}+\omega_0^2x=0$ $\omega_0=\sqrt{\frac{K}{m}}$ $\Delta F=K\Delta x$

$mg=K\,0.1$ $K=\frac{mg}{0.1}$

$\rightarrow F=Mg$

$\rightarrow F=\frac{m}{2}g$

$m=\frac{M}{2}$

$\omega_0=\sqrt{\frac{mg}{0.1m}}=\sqrt{\frac{g}{0.1}}=\sqrt{98.1}$

$\nu=\frac{\omega_0}{2\pi}=\frac{\sqrt{98}}{2\pi}=1.58\,\text{Hz}$

b) dissipation in suspension system. What is the decay time of traffic light amplitude oscillation after separation?

$\gamma\propto\frac{1}{m}$ $2m\rightarrow 4s$ $m\rightarrow 2s$

$\gamma=\frac{1}{m}$ because $\ddot{x}+\gamma\dot{x}+\omega_0^2x=\frac{F_0}{m}e^{i\omega t}$

c) Analytic expression for $y(t)$ after separation. Assume light was stationary ($\dot{y}=0$) at prev equilibrium position at time of separation $t=0$.

$\ddot{y}+\gamma\dot{y}+\omega_0^2y=0$ $y(t)=Ae^{-\frac{\gamma}{2}t}\cos(\omega_d t+\phi)$

$\frac{\gamma}{2}=\frac{1}{2s}=\frac{1}{2}s^{-1}$ $\omega_d=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}=\sqrt{98-(\frac{1}{2})}=9.887\,s^{-1}$

→ decay time $\Rightarrow e^{-\frac{\gamma}{2}t}$

$y(0)=-0.1=A\cos(\phi)$

$\dot{y}(t)=A-\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}\cos(\omega_d t+\phi)+$ $+A\omega_d-\sin(\omega_d t+\phi)=0$

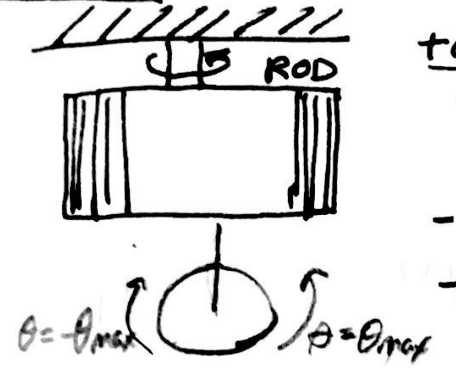
$\dot{y}(0)=-A\frac{\gamma}{2}\cos(\phi)-A\omega_d\sin(\phi)=0$

$-\frac{\gamma}{2\omega_d}=\tan\phi=-\frac{1s^{-1}}{2(9.887s^{-1})}$ $=A\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}\cos(\omega_d t+\phi)-A\omega_d\sin(\omega_d t+\phi)$

$\Rightarrow\phi=-2.89$ $A=\frac{-0.1}{\cos\phi}\approx\frac{-0.1\,m}{\cos(-2.89)}\approx-0.1\,m$

$\Rightarrow\boxed{y(t)=(-0.1)e^{-\frac{1}{2}t}\cos(9.887t+2.89)}$

Problem 4 : Torsional Oscillator as a Measurement Device



torsional oscillator

- hollow cylindrical spool hung by thin beryllium copper torsion rod from stationary pt. support
- Mylar film wound on spool
- avg spacing of 500 Å between layers of Mylar
$\Rightarrow$ allows gas to flow between sheets of film.
- frequency of oscillator : 1300Hz , $Q = 2.5\times10^5$

a) Find relation between fractional frequency shift $(d\nu/\nu)$ & fractional shift in mass of spool dM/M. "fractional" just proportion

angular with damping $\Rightarrow I\frac{d^2\theta}{dt^2} + C\frac{d\theta}{dt} + k\theta = \tau(t)$ C $\Rightarrow$ angular damping constant

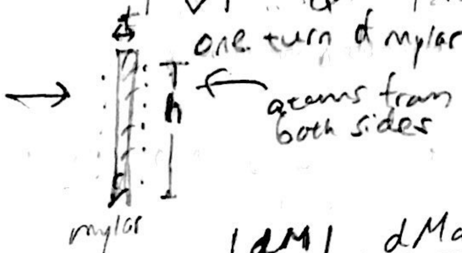
$\ddot{\theta} + \frac{C}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{\tau(t)}{I}$ $\quad I = \frac{1}{2}MR^2 \quad \tau = -K\theta = I\ddot{\theta} \quad \vec{L} = I\dot{\vec{\theta}} \quad \vec{\tau} = \frac{d\vec{L}}{dt} \quad \nu = \frac{\omega}{2\pi}$

w/o damping $I\ddot{\theta} = -K\theta$ $\quad \omega^2 = \frac{K}{I} \quad \Rightarrow \nu = \frac{\omega}{2\pi} \propto \sqrt{\frac{K}{I}} \propto \frac{1}{\sqrt{M}}$

$\nu = AM^{-1/2}$, $\frac{d\nu}{dM} = \left(-\frac{1}{2}\right)AM^{-3/2} = \left(-\frac{1}{2}\right)\nu M^{-1} \Rightarrow \boxed{\frac{d\nu}{\nu} = -\frac{1}{2}\frac{dM}{M}}$

b) mass of Mylar dominates mass of spool. Calculate change in # absorbed helium atoms / cm^2 resolvable Helium atomic weight = 4 mass of proton $= 1.67\times10^{-24}$ g
density of one layer helium atom = 10^{14} atoms/cm^2 (unimportant)
What fraction of a single layer does this resolution correspond to? pt 2 question
that can be resolved. Fractional changes in frequency of oscillator $10^{-3}/Q$ resolvable

$\left|\frac{d\nu}{\nu}\right| = \left|\frac{10^{-3}}{Q}\right| = \left|\frac{10^{-3}}{2.5\times10^5}\right| = 4\times10^{-9} \quad \left|\frac{dM}{M}\right| = \left|-2\frac{d\nu}{\nu}\right| = 8\times10^{-9}$



$M_{mylar} = 2\pi r h t \rho$ ← mass density of mylar

$M_{atoms} = 2(2\pi r h) m \sigma$ ← surface density $\frac{atoms}{cm^2}$

two sides; mass of one atom

$\left|\frac{dM}{M}\right| = \frac{dM_{atoms}}{M_{mylar}} = \frac{2(2\pi r h) m d\sigma}{2\pi r h t \rho} = \frac{2 m d\sigma}{t\rho} = \frac{2(1.67\times10^{-24}\times4)}{(2.5\times10^{-4})\times1.42}d\sigma$ atoms, cm

$= 3.76\times10^{-20}d\sigma = 8\times10^{-9} \Rightarrow d\sigma = 2\times10^{11}$ (Atoms/cm^2)

Fraction of monolayer $= \frac{d\sigma}{\sigma} = \frac{2\times10^{11}}{10^{14}} = 2\times10^{-3}$

unit checking: $\underbrace{\frac{2(g \times \text{atomic weight})}{cm \times \frac{g}{cm^3} = \frac{g}{cm^2}}}_{cm^2 \text{ atomic weight}} \times \frac{atoms}{cm^2}$

cm^2 atomic weight $\times \frac{atoms}{cm^2} \Rightarrow$ mass? $= \frac{dM}{M}$

12/25/19 Pset 2

problem 5: Addition of Harmonic Variables

a) - linear system driven by F_1 oscillates w/ amplitude 2.000 cm
- driven by another force F_2 at same frequency, amplitude is 1.414 cm
- driven by both forces at same time: $F = F_1 + F_2$, amplitude is 1.414 cm
$\Rightarrow$ what is the phase difference between F_1 & F_2?

irrelevant to problem but critical concept

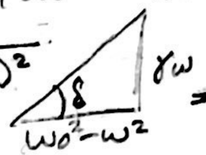
$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$ guess sol'n $z(t) = Ae^{i(\omega t + \delta)}$ $\boxed{\text{background}}$

$P(D) = D^2 + \gamma D + \omega_0^2$ $P(i\omega) = -\omega^2 + i\gamma\omega + \omega_0^2$ $\omega \neq \omega_0$ is the drive frequency not natural freq.

$(-\omega^2 + i\gamma\omega + \omega_0^2) Ae^{i(\omega t + \delta)} = \frac{F_0}{m} e^{i\omega t}$

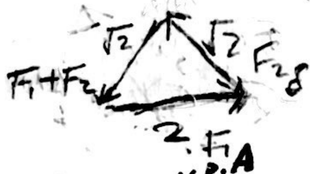
divide both sides by $e^{i(\omega t + \delta)}$ $\Rightarrow (\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m} e^{-i\delta}$ phase argument

$Ae^{i\delta} \Rightarrow$ 

$\sqrt{(\omega_0 - \omega)^2 + (\gamma\omega)^2}$  $\Rightarrow A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0 - \omega)^2 + (\gamma\omega)^2}}$

$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$

$F = F_1 + F_2 \Rightarrow X = X_1 + X_2$



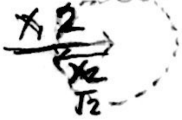
$\delta \equiv \boxed{180 - 45 = 135°}$

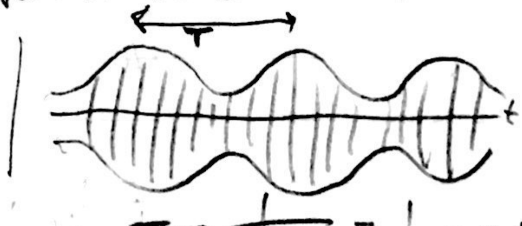
b) Magnitudes same as in A, but now frequencies of F_1 & F_2 are $\nu_1 = 100\text{Hz}$ & $\nu_2 = 102\text{Hz}$
ν_1 & ν_2 not near natural frequencies of system.
Make sketch of system to sum of two forces $F = F_1 + F_2$
Sketch amplitude vs. time ~~the~~ after steady-state.
Give max. & min. amplitudes & peak to peak separation

$\vec{X} = \vec{X}_1 + \vec{X}_2$

max  $\longrightarrow 2 + \sqrt{2} = 3.414$ cm

amplitude vs. time after steady-state

min  $\longrightarrow 2 - \sqrt{2} = 0.586$ cm

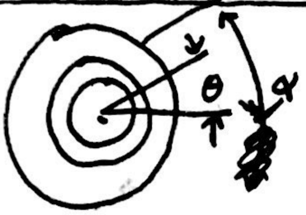


peak-to-peak separation

$T = \frac{1}{\nu} = \frac{1}{2}$

$T = \frac{1}{|\nu_1 - \nu_2|} = \frac{1}{2} = 0.5$ s

Problem 6: Driven Torsional Oscillator



- torsional oscillator w/ moment of inertia I driven by viscous drag between it & outer concentric ring
- restoring torque: $-K\theta$ torque due to friction: $-b\frac{d}{dt}(\theta-\alpha)$
  b: constant α: angular pos. of outer ring
- outer ring driven in steady state ~~such~~ $\alpha(t) = \alpha_0 \cos(\omega t)$

a) Write down d.f.q. governing angular position $\theta(t)$ of oscillator.

is α $\dot{\theta}$?

$I\ddot{\theta}(t) + b\dot{\theta}(t) + K\theta =$ $\quad I\ddot{\theta} = -K\theta - b\frac{d}{dt}(\theta-\alpha)$

$I\ddot{\theta} + b\dot{\theta} + K\theta = b\dot{\alpha}$ $\quad = -K\theta - b\dot{\theta} + b\dot{\alpha}$ $\quad \alpha = \alpha_0\cos(\omega t)$

$$\boxed{\ddot{\theta} + \frac{b}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{b}{I}\dot{\alpha} = \frac{b}{I}\frac{d}{dt}(\alpha_0\cos(\omega t))}$$

b) The angular displacement has form: $\theta(t) = \theta_0(\omega)\cos(\omega t - \delta(\omega))$
Find $\theta_0(\omega)/\alpha_0$ & make sketch of result.

$\theta(t) = \theta_0(\omega)\cos(\omega t - \delta(\omega))$

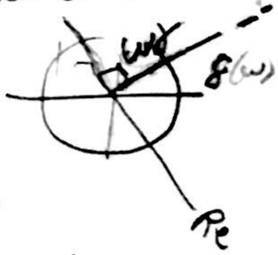
$\theta(t) = Re[\theta_0(\omega) e^{i(\omega t - \delta(\omega))}]$ $\quad \alpha = Re[\alpha_0 e^{i\omega t}]$

$\ddot{\theta} + \frac{b}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{b}{I}\frac{d}{dt}[Re\,\alpha_0 e^{i\omega t}] = Re[\frac{b}{I}\alpha_0 i\omega e^{i\omega t}] = Re[\gamma\alpha_0 i\omega e^{i\omega t}]$

$P(D) = D^2 + \frac{b}{I}D + \frac{K}{I}$ $\quad P(i\omega) = -\omega^2 + \gamma i\omega + \underbrace{\frac{K}{I}}_{\omega_0^2}$

$(-\omega^2 + i\gamma\omega + \omega_0^2)\theta_0(\omega)\frac{e^{i(\omega t - \delta(\omega))}}{e^{i(\omega t - \delta(\omega))}} = i\gamma\alpha_0\omega\frac{e^{i\omega t}}{e^{i(\omega t - \delta(\omega))}}$

$\Rightarrow (-\omega^2 + i\gamma\omega + \omega_0^2)\theta_0(\omega) = i\omega\gamma\alpha_0 e^{i\delta(\omega)}$

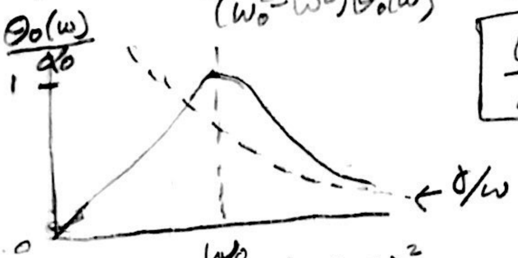


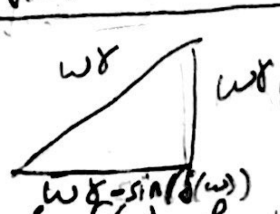
$= \sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}\,\theta_0$ $\quad \gamma\omega\theta_0(\omega)$ $\quad (\omega_0^2-\omega^2)\theta_0(\omega)$

$\frac{\theta_0(\omega)}{\alpha_0} = \frac{|i\omega\gamma e^{i\delta(\omega)}|}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}$

$$\boxed{\frac{\theta_0(\omega)}{\alpha_0} = \frac{\omega\gamma}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}}$$

explanation





$\Leftarrow$ $e^{i\delta(\omega)} = \cos(\delta(\omega)) + i\sin(\delta(\omega))$
$ie^{i\delta(\omega)} = i\cos(\delta(\omega)) - \sin(\delta(\omega))$

c) Find analytical expression for phase angle $\delta(\omega)$ & sketch result.

$\frac{-\omega\gamma\alpha_0\sin\delta}{\omega\gamma\alpha_0\cos\delta} = \frac{(\omega_0^2-\omega^2)\theta_0(\omega)}{\omega\gamma\theta_0(\omega)}$

Im: $\omega\gamma\theta_0(\omega) = \omega\gamma\alpha_0\cos\delta$
Re: $(\omega_0^2-\omega^2)\theta_0(\omega) = -\omega\gamma\alpha_0\sin\delta$

$\Rightarrow \tan\delta = -\frac{\omega_0^2-\omega^2}{\omega\gamma}$