Lecture 8

From previous lecture

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \hat{H}(x)\psi(x,t) = \left[\frac{\hat{p}^2}{2m} + V(x)\right]\psi(x,t)$$

$$\psi(x,t) = e^{\frac{-iEt}{\hbar}}\psi(x)$$

$$E\psi(x) = \hat{H}(x)\psi(x) = \left[\frac{\hat{p}^2}{2m} + V(x)\right]\psi(x)$$

Eigenfunction expansion $\boxed{\psi(x,t) = \sum_i a_i \phi_i(x) e^{-iE_i t/\hbar}}$

Orthogonality

$$\int_{-\infty}^{\infty} \phi_k^*(x)\phi_j(x)dx = \langle \phi_k | \phi_j \rangle = 0 \quad \text{if } E_j \neq E_k$$

If $E_j = E_k$ can form orthogonal ϕ_j, ϕ_k using linear superposition

orthogonal - eigenfunctions $\langle \phi_k | \phi_j \rangle = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$

Initial condition

$$\psi(x,0) = \sum_j a_j \phi_j(x) \Rightarrow a_j = \langle \phi_j(x) | \psi(x,0) \rangle$$

$$= \int_{-\infty}^{\infty} \phi_j^*(x)\psi(x,0)\,dx$$

Completeness

$$\sum_i |\phi_i\rangle\langle\phi_i| = 1$$

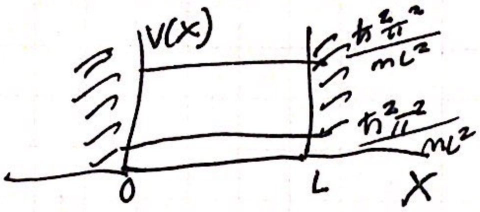
alternatively

$$f(x) = \sum_j \int \phi_j^*(x')f(x')dx'\,\phi_j(x) = \int_{-\infty}^{\infty} \underbrace{\sum_j \phi_j^*(x')\phi_j(x)}_{\delta(x-x')} f(x')dx'$$

to a complete basis

$K_n = \frac{n\pi}{L}$

$E\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)$



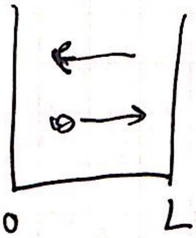
$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) = \sqrt{\frac{2}{L}}\,\frac{e^{iK_n x} - e^{-iK_n x}}{2i}$

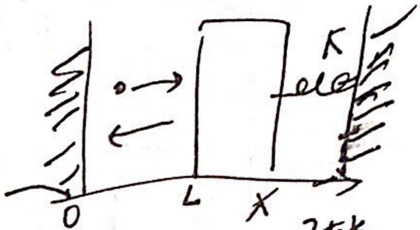
$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$

Classical analog

$E_n = \frac{p_n^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}n^2 \quad p_n = mv_n = \hbar k_n = \hbar\left(\frac{n\pi}{L}\right)$

$p_n = mv_n = \frac{\hbar\pi n}{L}$





$F = \frac{-d}{dL}\left(\frac{\hbar\pi^2}{2mL^2}n^2\right) = \frac{\hbar^2\pi^2 n^2}{mL^3}$

$\parallel$

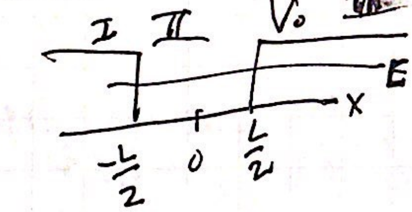
E_n

$F_n = \frac{\Delta P_n}{\Delta t_n} \quad \Delta P_n = 2P_n = 2\frac{\hbar\pi n}{L} = 2\hbar k_n$

$\Delta t_n = \frac{2L}{V_n} = \frac{2L}{\frac{\hbar\pi n}{mL}}$

$= \frac{2\hbar k_n}{\frac{2Lm}{\hbar k_n}} = \frac{(\hbar k_n)^2}{mL} = \frac{\hbar^2\pi^2 n^2}{mL^3}$

Finite well



Allowed

$E\psi_{II} = \frac{-\hbar^2}{2m}\frac{d}{dx^2}\psi_{II}$

$\psi_{II}(x) = e^{ikx},\, e^{-ikx}$

$\cos kx, \sin kx$

Forbidden

$E\psi_{III} = \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\psi_{III}$

$\psi_{III} = e^{\alpha x}, e^{-\alpha x}$

$\cosh(\alpha x), \sinh(\alpha x)$

Symmetric

$\psi_I(x) = ae^{\beta x}$

$\psi_{II}(x) = \cos(kx)$

$\psi_{III}(x) = ae^{-\beta x}$

$\psi_{II}\Big|_{\frac{L}{2}} = \psi_{III}\Big|_{\frac{L}{2}} \Rightarrow \cos\left(\frac{kL}{2}\right) = ae^{-\frac{\beta L}{2}}$

$\frac{d\psi_{II}}{dx}\Big|_{\frac{L}{2}} = \frac{d\psi_{III}}{dx}\Big|_{\frac{L}{2}} \Rightarrow -k\sin\left(\frac{kL}{2}\right) = -\beta ae^{-\frac{\beta L}{2}}$

$\frac{\sin\left(\frac{kL}{2}\right)}{\cos\left(\frac{kL}{2}\right)} = \frac{\beta}{k} \Rightarrow \tan\left(\sqrt{\frac{2mE}{\hbar^2}}\frac{L}{2}\right) = \sqrt{\frac{V_0 - E}{E}}$

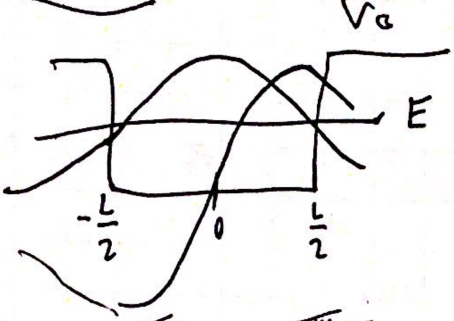
What is this constraint telling me?

$$E\psi(x) = \frac{-\hbar^2}{2m}\frac{d}{dx}^2\psi(x) + V(x)\psi(x)$$

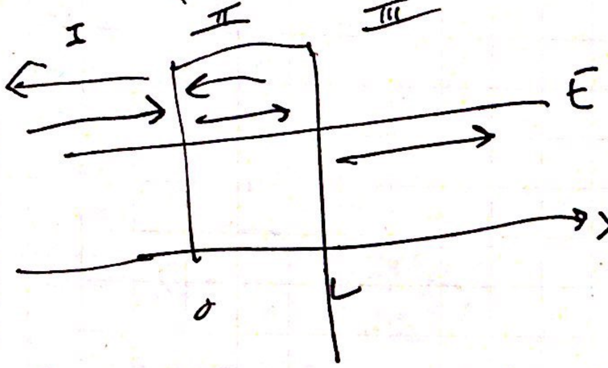
$$\Psi(x,t) = \sum_j a_i \phi_j(x) e^{-iE_j t/\hbar}$$

Solutions

$\langle \frac{\hat{p}^2}{2m} \rangle$



We'll come back to this on Wednesday.



$$\psi_I(x) = e^{ikx} + re^{-ikx}$$

$$\psi_{II}(x) = ae^{\gamma x} + be^{-\gamma x}$$

$$\psi_{III}(x) = te^{ika}$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$E = \frac{-\hbar\gamma^2}{2m} + V_0$$

$$E\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)$$

at $x=0$ $\quad 1+r = a+b$

$\psi_I = \psi_{II}$

$\psi_I' = \psi_{II}'$ $\quad ik(1-r) = \gamma(a-b)$

$$e^{ikL} + re^{-ikL} = ae^{\gamma L} + be^{-\gamma L}$$

Trick will help for pset

$$ik(e^{ikL} - re^{-ikL}) = \gamma(ae^{\gamma L} - be^{-\gamma L})$$

$$ae^{\gamma L} + be^{-\gamma L} = ikte^{ikL}$$

$$\gamma(ae^{\gamma L} - be^{\gamma L}) = ikte^{ikL}$$

$$\frac{a}{t}e^{\gamma L} + \frac{b}{t}e^{-\gamma L} = e^{ikL}$$

$$\frac{a}{t}e^{\gamma L} - \frac{b}{t}e^{-\gamma L} = \frac{ik}{\gamma}e^{ikL}$$

☆ $\frac{1}{t} + \frac{r}{t} = \frac{a}{t} + \frac{b}{t}$ $\qquad \frac{2}{t} = \frac{a}{t}\left(1 + \frac{\gamma}{ik}\right) + \frac{b}{t}\left(1 - \frac{\gamma}{ik}\right)$

$\frac{1}{t} - \frac{r}{t} = \frac{\gamma}{ik}\left(\frac{a}{t} - \frac{b}{t}\right)$

$$2\frac{a}{t}e^{\gamma L} = \left(1 + \frac{ik}{\gamma}\right)e^{ikL}$$

$$2\frac{b}{t}e^{-\gamma L} = \left(1 - \frac{ik}{\gamma}\right)e^{ikL}$$