Why can you do this with plane waves?
(it's because they're orthogonal)

Lecture 7 9/18

From last time

$$\int_{-\infty}^{\infty} (\hat{H}\Psi)^* \hat{Q}\,\Psi\, dx = \int_{-\infty}^{\infty} \Psi^* \hat{H}\hat{Q}\Psi\, dx$$

then H, Ψ are hermitians (really important) property

Ehrenfest Theorem

$$\frac{d}{dt}\langle \hat{Q}\rangle = \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle [\hat{Q}, \hat{H}]\rangle$$

— Can use to get dynamical classical version of quantum model

time-dependent Schrodinger eq'n

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}(x)\,\Psi(x,t) = \left[\frac{\hat{p}^2}{2m} + V(x)\right]\Psi(x,t)$$

periodic sol'n: $\Psi(x,t) = \Psi(x)e^{-iEt/\hbar}$

$$E\,\Psi(x) = \hat{H}(x)\,\Psi(x) = \left[\frac{\hat{p}^2}{2m} + V(x)\right]\Psi(x)$$

← Time independent Schrödinger eq'n

$$E_i\,\phi_j(x) = \hat{H}(x)\,\phi_j(x) = \left[\frac{\hat{p}}{2m} + V(x)\right]\phi_j(x)$$

$$\boxed{\Psi(x,t) = \sum_i a_j\,\phi_j(x)\,e^{-\frac{iE_t}{\hbar}}}$$

both are expansion coefficient

Free Space:

$$\Psi(x,t) = \int_{-\infty}^{\infty} A(k)\,e^{ikx}e^{-i\omega(k)t}\,\frac{dk}{2\pi}$$

$E=\hbar\omega$
could rewrite as $\frac{E(k)}{\hbar}$

$$\Psi(x,0) = \int_{-\infty}^{\infty} A(k)\,e^{ikx}\,\frac{dk}{2\pi}$$

$$\int_{-\infty}^{\infty} e^{-ikx}\Psi(x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(k)e^{ikx}\,\frac{dk}{2\pi}\,e^{-ikx}\,dk$$

$$= \int_{-\infty}^{\infty} A(k) \underbrace{\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx}_{2\pi\,\delta(k-k')} \frac{dk}{2\pi}$$

$$= A(k')$$

development of the generalization of the fourier transform ⟹ Can we get this with the eigenfunction expansion?

$$\Psi(x,0)=\sum_j a_j\,\phi_j(x)$$

if orthogonality works, $\int_{-\infty}^{\infty}\phi_k^*(x)\,\psi(x,0)\,dx=\int_{-\infty}^{\infty}\sum_j a_j\,\phi_j(x)\,\phi_k^*(x)\,dx$

$$=\boxed{\sum_j a_j\underbrace{\int_{-\infty}^{\infty}\phi_k^*(x)\,\phi_j(x)\,dx}_{}=a_k}$$

$=1$ when $j=k$

0 otherwise

$$\int_{-\infty}^{\infty}\phi^*_k(x)\,\phi_j(x)\,dx=\begin{cases}0 & j\neq k\\ 1 & j=k\end{cases}$$

Generalization of the fourier transform?

$$E_j\phi_j=\hat{H}\phi_j \qquad E_j^*\phi_j^*=\hat{H}^*\phi_j^*$$

$$\Rightarrow E_j\int\phi_j^*(x)\,\phi_j(x)\,dx=\int_{-\infty}^{\infty}\phi_j^*(x)\,\hat{H}\,\phi_j(x)\,dx$$

$$\Rightarrow E_j\int\phi_j^*(x)\,\phi_j(x)\,dx=\underbrace{\int_{-\infty}^{\infty}H^*\phi_j^*\phi_j\,dx}_{\text{H is a hermacian!}}=\int_{-\infty}^{\infty}\phi_j^*(x)\,\hat{H}\,\phi_j(x)\,dx$$

$\Rightarrow E_j=E_j^*$ they're real!

all the energy eigenvalues are real then the problem is a hermacian.

Quantum mechanics	Linear Algebra
$\hat{H}\,\phi_j=E_j\,\phi_j$ $\Leftrightarrow$	$Av=\lambda v$
$\hat{H}$: operator, E_j: eigenvalue	A: matrix, λ: eigenvalue

$$E_j\phi_j=\hat{H}\phi_j$$

$$E_k\phi_k^*=\hat{H}^*\phi_k^*$$

$$\Rightarrow E_j\int_{-\infty}^{\infty}\phi_k^*(x)\,\phi_j(x)\,dx=\int_{-\infty}^{\infty}\phi_k^*\hat{H}\,\phi_j\,dx$$

$$E_k\int_{-\infty}^{\infty}\phi_k^*\phi_j\,dx=\int_{-\infty}^{\infty}(\hat{H}\phi_k)^*\phi_j\,dx=\int_{-\infty}^{\infty}\phi_k^*\hat{H}\,\phi_j\,dx$$

If these are different, why is the result same? $\Rightarrow$ inner product must be 0

Conclusion:

if $E_j\neq E_k$ Then $\int_{-\infty}^{\infty}\phi_k^*(x)\,\phi_j(x)\,dx=0$

☆ The eigenfunctions are orthogonal if their energies are different

- If they're the same, no guarantee of orthogonality if $E_j \neq E_k$, then can get ϕ_j^*, ϕ_k^* to be orthogonal

$$\int_{-\infty}^{\infty} \phi_j^* \phi_j \, dx = 1$$

→ normalization.

$$a_k = \int_{-\infty}^{\infty} \phi_k^*(x) \, \Psi(x,0) \, dx$$

$$a_k = \frac{\int_{-\infty}^{\infty} \phi_k^*(x) \, \Psi(x,0) \, dx}{\int_{-\infty}^{\infty} |\phi_k(x)|^2 dx}$$

orthogonality's going to be pretty important

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} \frac{dk}{2\pi} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right] e^{ikx} \frac{dk}{2\pi}$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx'} dx' = \int_{-\infty}^{\infty} dx' \left[\underbrace{\int_{-\infty}^{\infty} e^{ik(x-x')} \frac{dk}{2\pi}}_{\delta(x-x')} f(x') \right]$$

$$\delta(x-x') = \int_{-\infty}^{\infty} e^{ikx} e^{-ikx} \frac{dk}{2\pi} \qquad = \int_{-\infty}^{\infty} f(x') \delta(x-x') \, dx' = f(x)$$

Completeness

$$f(x) = \sum_j a_j \phi_j(x) = \sum_j a_j |\phi_j\rangle \qquad |\phi_j\rangle = \phi_j(x)$$

this is how the notation works

$$a_j = \int_{-\infty}^{\infty} \phi_j^*(x) f(x) dx = \langle \phi_j | f(x) \rangle \qquad \langle \phi_j | \doteq \int_{-\infty}^{\infty} \phi_j^*(x) \, \underline{\quad\quad} \, dx$$

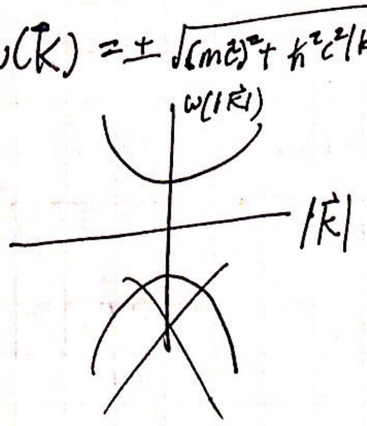
$$f(x) = \sum_j |\phi_j\rangle a_j \qquad f(x) = \left(\sum_j |\phi_j\rangle\langle\phi_j| \right) f(x)\rangle \qquad \sum_j |\phi_j\rangle\langle\phi_j| = 1$$

this expansion is called the Hilbert expansion

$$\hat{H} = \frac{\hat{p}^2}{2m} + V$$

do we have completeness?

$$\hbar\omega(\vec{k}) = \pm\sqrt{(mc^2)^2 + \hbar^2 c^2 |k|^2}$$

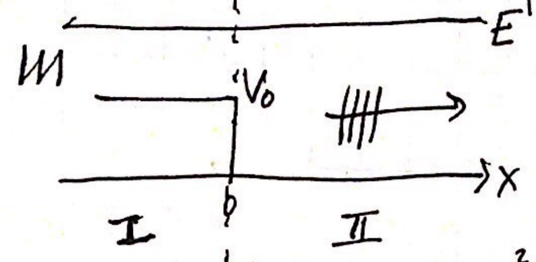


Interested in solutions...

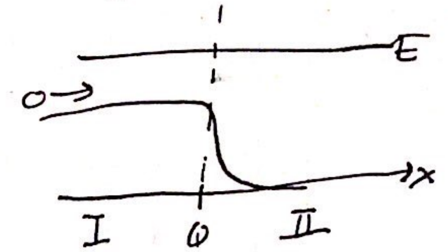
$$E\Psi(x) = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \Psi(x) + V(x)\Psi(x)$$

$\approx$ 20 important problems where exact sol'ns can be developed

Quantum mechanical problem



classical



$$E = \frac{(\hbar k)^2}{2m} + V_0 \qquad E = \frac{(\hbar k_2)^2}{2m}$$

$$E = \frac{P_i^2}{2m} + V_0 \qquad E = \frac{P_f^2}{2m}$$

$$\psi_I(x) = e^{ik_i x} + r e^{-ik_i x}$$

$$\psi_{II}(x) = t e^{ik_f x}$$

Boundary Conditions at $x=0$

$$\psi_I(x=0) = \psi_{II}(x=0) \Rightarrow 1+r=t$$

$$\left.\frac{d\psi}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0} = ik(1-r) = ik_f t$$

$$1+r=t$$

$$1-r = \frac{k_t}{k_i} t$$

$$2 = \left(1+\frac{k_t}{k_i}\right)t \Rightarrow t = \frac{2}{1+\frac{k_t}{k_i}}$$

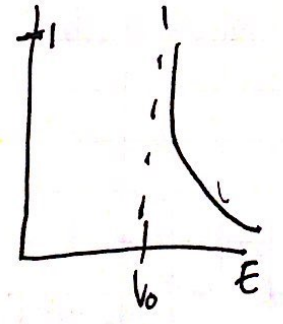
$$2r = t\left(1-\frac{k_t}{k_i}\right) \qquad r = \frac{1-\frac{k_t}{k_i}}{1+\frac{k_t}{k_i}} = \frac{k_i - k_t}{k_i + k_t}$$

$$R = |r|^2 = \frac{|k_i - k_t|^2}{|k_i + k_t|^2}$$

$$= \left|\frac{\sqrt{\frac{2m(E-V_0)}{\hbar^2}} - \sqrt{\frac{2mE}{\hbar^2}}}{\sqrt{\frac{2m(E-V_0)}{\hbar^2}} + \sqrt{\frac{2mE}{\hbar^2}}}\right|^2$$

$$R = \left|\frac{\sqrt{E-V_0} - \sqrt{E}}{\sqrt{E-V_0} + \sqrt{E}}\right|^2$$

R(E)



What needs to be true for this result to work?

? Question

has to do w/ the speed of light on both sides. probability flux instead of power flow

$$T = 1 - R = 1 - \frac{(k_i - k_t)^2}{(k_i + k_f)^2} = \frac{(k_i+k_f)^2 - (k_i - k_t)^2}{(k_i + k_t)^2} = \frac{4k_i k_t}{(k_i + k_t)^2}$$

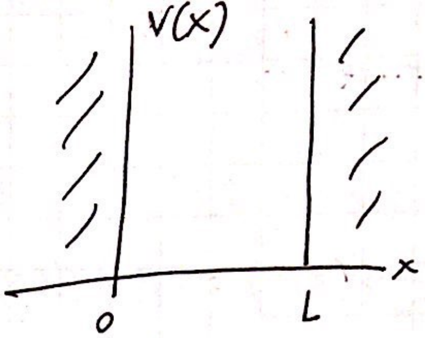
$$= \frac{k_f}{k_i}|t|^2$$

$$|t|^2 = \frac{4k_i^2}{(k_i+k_f)^2} \qquad T \neq |t|^2$$

$J_i \sim k|1|^2 \quad p = \hbar k$

$J_r \sim k_i |r|^2 \quad \overset{\parallel}{mv} \Rightarrow v \sim \frac{\hbar k}{m}$

$J_t \sim k_f |t|^2$



$\phi_n(x) = C\sin\left(\frac{n\pi x}{L}\right)$

$E\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)$

plug this into

$0 \le x \le L, \quad E\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)$

$\Rightarrow$

$\Rightarrow E\sin\left(\frac{n\pi x}{L}\right) = \frac{-\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 \sin(\,)$

etc.

$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

$\int_0^L |C|^2 \sin^2(\quad)\,dx = 1$

$\frac{1}{2}L|C|^2 = 1$