Lecture 6

From last time

$$\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\int_{-\infty}^{\infty} x\,\Psi^*(x,t)\Psi(x,t)\,dx$$

$$=\int_{-\infty}^{\infty} x\,\frac{\partial \Psi^*}{\partial t}\Psi\,dx + \int_{-\infty}^{\infty} x\,\Psi^*\frac{\partial \Psi}{\partial t}\,dx$$

$$= \dots = \left\langle \frac{-i\hbar}{m}\frac{\partial}{\partial x}\right\rangle = \frac{\langle \hat{p}\rangle}{m}$$

Operators

	k
$\hat{p} = -i\hbar\frac{\partial}{\partial x}$	$\hbar k$
$\hat{x} = x$	$\frac{i\partial}{\partial k}$
$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$	$\frac{\hbar^2 k^2}{2m}$
$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+V$	$\frac{\hbar^2 k^2}{2m}+\hat{V}$

$$\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\frac{-i\hbar}{m}\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx$$

$$\boxed{\frac{d}{dt}\langle \hat{Q}\rangle = \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle [\hat{Q},\hat{H}]\rangle}$$

$$[\hat{Q},\hat{H}] = \hat{Q}\hat{H} - \hat{H}\hat{Q}$$

$$\int_{-\infty}^{\infty}(\hat{H}\Psi)^*\hat{Q}\Psi\,dx = \int_{-\infty}^{\infty}\Psi^*\hat{H}\hat{Q}\,dx \Rightarrow \text{ if } \hat{H} \text{ is Hamiltonian}$$

(definition of Hamiltonian)

$$\frac{d}{dt}\langle x\rangle = \frac{\langle \hat{p}\rangle}{m} \quad \Leftrightarrow \quad \frac{d}{dt}x(t) = \frac{p(t)}{m}$$

$$\frac{d}{dt}\langle \hat{p}\rangle = -\left\langle \frac{\partial V}{\partial x}\right\rangle \qquad \frac{d}{dt}p(t) = \frac{-dV}{dt}$$

$$\langle \hat{Q}\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat{Q}\,\Psi(x,t)\,dx = \langle \Psi(x,t)|\hat{Q}|\Psi(x,t)\rangle$$

$$|\Psi(x,t)\rangle = \Psi(x,t)$$

$$\langle \Psi(x,t)| = \int_{-\infty}^{\infty}\Psi(x,t)\,\underline{\quad}\,dx$$

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,P(x,t)\,dx$$

$$= \int_{-\infty}^{\infty} x\,\Psi^*(x,t)\,\Psi(x,t)\,dx = \int_{-\infty}^{\infty}\Psi^*(x,t)\,x\,\Psi(x,t)\,dx = \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx$$

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Fourier transform it

$$= \int_{-\infty}^{\infty} A^*(k,t)\,\hbar k\, A(k,t)\,\frac{dk}{2\pi} \longrightarrow \int_{-\infty}^{\infty} \hbar k\, P_k(k,t)\,dk$$

$$\langle 1\rangle = \int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx \qquad \frac{d}{dt}\langle 1\rangle = \langle \cancel{\tfrac{\partial 1}{\partial t}}\rangle + \frac{1}{i\hbar}\langle \cancel{[1,\hat H]}\rangle$$

$$[1,\hat H] = 1\cdot\hat H - H\cdot 1 = 0 \qquad \hat H = \frac{\hat p^2}{2m} + V(x)$$

If prob. is Hermitian (?) probably is conserved

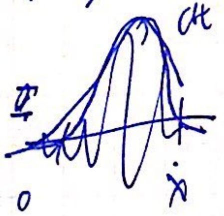
If not, then probability isn't even conserved $\quad [\hat H,\hat H] = \hat H\hat H - \hat H\hat H = 0$

$$\frac{d}{dt}\langle \hat H\rangle = \langle \cancel{\tfrac{\partial \hat H}{\partial t}}\rangle + \frac{1}{i\hbar}\langle[\hat H,\hat H]\rangle = 0$$

energy is conserved as long as potential & mass not time dependent

$$\frac{d}{dt}\langle x\rangle = \langle\cancel{\tfrac{\partial x}{\partial t}}\rangle + \frac{1}{i\hbar}\langle[x,\hat H]\rangle \qquad \frac{d}{dt}\int_{-\infty}^{\infty}\Psi^*(x,t)\,x\,\Psi(x,t)\,dx$$

$$\langle v\rangle = \frac{d}{dt}\langle x\rangle \quad \frac{\langle\hat p\rangle}{m} \quad \langle\hat p\rangle = m\langle v\rangle \qquad = \int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial t}\,x\,\Psi\,dx + \int \Psi^*\left(\frac{\partial}{\partial t}x\right)\Psi\,dx + \int \Psi^* x\frac{\partial\Psi}{\partial t}\,dx$$



$$\frac{d}{dt}\langle x\rangle = \frac{1}{i\hbar}\left\langle\left[x,\frac{\hat p^2}{2m}+V\right]\right\rangle$$

$$= \frac{1}{i\hbar}\left\langle\left[x,\frac{\hat p^2}{2m}\right]\right\rangle + \frac{1}{i\hbar}\langle\cancel{[x,V]}\rangle \qquad \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\right)\left\langle -2\frac{\partial}{\partial x}\right\rangle = \frac{\langle -i\hbar\frac{\partial}{\partial x}\rangle}{m} = \frac{\langle\hat p\rangle}{m}$$

$$[x,V(x)] = xV(x) - V(x)x = 0$$

$$\left[x,\frac{\hat p^2}{2m}\right] = \left[x, -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right] = -\frac{\hbar^2}{2m}\left[x,\frac{\partial^2}{\partial x^2}\right] = -\frac{\hbar^2}{2m}\left(x\frac{\partial^2}{\partial x^2} - \frac{\partial^2 x}{\partial x^2}\right)$$

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$$\left[x, \frac{d^2}{dx^2}\right] f(x) = x\frac{d^2}{dx^2}f - \frac{d^2}{dx^2}(xf) = x\frac{d^2}{dx^2}f - \frac{d}{dx}\left(f + x\frac{df}{dx}\right)$$

$$= x\frac{d^2 f}{dx^2} - \frac{df}{dx} - \frac{df}{dx} - \cancel{x\frac{d^2 f}{dx^2}}$$

$$\left[x, \frac{d^2}{dx^2}\right] = -2\frac{d}{dx}$$

$$= -2\frac{d}{dx}f(x)$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left|\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right|\psi(x,t) = \hat{H}(x)\psi(x,t)$$

$$\psi(x,t) = X(x)\,T(t)$$

$$\frac{i\hbar\frac{\partial}{\partial t}(XT)}{XT} = \frac{\hat{H}(x)(XT)}{XT} \rightarrow \frac{i\hbar\frac{\partial}{\partial t}T(t)}{T(t)} = \frac{\hat{H}(x)X(x)}{X(x)} = E$$

equations only involving time & x equal? ~~in space is equal to space~~

so it must be a constant

$\downarrow$

(E)

$$i\hbar\frac{d}{dt}T(t) = E\,T(t) \Rightarrow$$

$$T(t) = \text{const}\ e^{-\frac{iEt}{\hbar}}$$

$$E\,X(x) = \hat{H}(x)X(x) = \left[\frac{\hat{p}^2}{2m} + V(x)\right]X(x)$$ time independent Shrödinger equation

$$\psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

$$E\psi(x) = \hat{H}\psi(x) \rightarrow E_i\phi_j(x) = \hat{H}\phi_j(x)$$

Free space:

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k)\,e^{ikx}e^{-i\frac{E(k)t}{\hbar}}\frac{dk}{2\pi} \qquad \psi(x,0) = \int_{-\infty}^{\infty} A(k)\,e^{ikx}\frac{dk}{2\pi}$$

$$A(k) = \int_{-\infty}^{\infty}\psi(x,0)\,e^{-ikx}dx$$

general

$$\psi(x,t) = \sum_i a_i\,\phi_j(x)\,e^{-iE_it/\hbar}$$

$$\psi(x,0) = \sum_j a_j\,\phi_j(x) \qquad a_j = \langle\psi, |\psi(x,0)\rangle$$

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