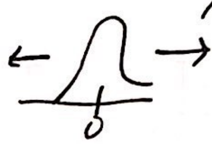
Lecture 5

Uncertainty ~~principle~~ relation $\Delta x \Delta p \geq \frac{\hbar}{2}$

$P_{\bar{X}}(x,t)$

$= |\Psi(x,t)|^2$

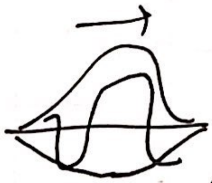


$p = \hbar k$

$P_k(k) = \frac{|A(k)|^2}{2\pi}$

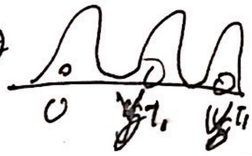
Moving Wave Packet

$$\Psi(x,0) = \frac{1}{(\pi L^2)^{1/4}} e^{ik_0 x} e^{-x^2/2L^2}$$



$\Rightarrow A(k) = (4\pi L^2)^{1/4} e^{-\frac{(k-k_0)^2 L^2}{2}}$

$\Rightarrow$



Relativistic Case

$$\Psi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx} e^{-i\omega(k)t} \frac{dk}{dt}$$

$$E = \sqrt{(mc^2)^2 + c^2 p^2}$$

$$\hookrightarrow \hbar\omega(k) = \sqrt{(mc^2)^2 + c^2\hbar^2 k^2}$$

Maxwell eqns

$\omega = \pm c|\vec{k}|$

dispersion relation

quantum $\hbar\omega = \hbar c|\vec{k}|$

Schrödinger eqn' in free space

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

Schrödinger equation for particle in a potential

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\,\psi(x,t)$$

(we don't understand how schrödinger's eq'n came about)

⊕ they're trying to fit ~~the~~ schrödinger's eq'n to $\langle p\rangle/m$

$$\frac{d}{dt}x(t) = \frac{p(t)}{m} \Rightarrow \frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}$$

$$\frac{d}{dt}p(t) = -\frac{d}{dx}V(x(t)) \qquad \frac{d}{dt}\langle p\rangle = -\left\langle \frac{d}{dx}V\right\rangle$$

$$\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\int_{-\infty}^{\infty} x\,\psi^*(x,t)\,\psi(x,t)\,dx = \int_{-\infty}^{\infty} x\frac{\partial\psi^*}{\partial t}\psi\,dx + \int_{-\infty}^{\infty} x\psi^*\frac{\partial\psi}{\partial t}\,dx$$

$$\frac{\partial}{\partial t}\psi = \frac{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi}{i\hbar} \qquad \frac{\partial}{\partial t}\psi^* = \frac{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^* + V\psi^*}{i\hbar}$$

$$\frac{d}{dt}\langle x\rangle = \int_{-\infty}^{\infty} x\left|\frac{-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \cancel{V\psi^*}}{-i\hbar}\right|\psi\,dx + \int_{-\infty}^{\infty} x\left|\frac{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \cancel{V\psi}}{i\hbar}\right|\psi^*\,dx$$

$$\frac{d}{dt}\langle x\rangle = \frac{\hbar}{2mi}\int_{-\infty}^{\infty} x\frac{\partial^2\psi^*}{\partial x^2}\psi\,dx - \frac{\hbar}{2mi}\int_{-\infty}^{\infty} x\,\psi^*\frac{\partial^2\psi}{\partial x^2}\,dx \overset{?}{=} \frac{\langle p\rangle}{m}$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \qquad \int_{-\infty}^{\infty} x\frac{\partial^2\psi^*}{\partial x^2}\psi\,dx \qquad u = x\psi \quad dv = \frac{\partial^2\psi^*}{\partial x^2}dx$$

$$x\psi\frac{\partial\psi^*}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\frac{\partial}{\partial x}(x\psi)\,dx$$

$$\int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\frac{\partial}{\partial x}(x\psi)\,dx$$

$$\frac{d}{dt}\langle x\rangle = \frac{\hbar}{2mi}\int_{-\infty}^{\infty}\psi^*\frac{\partial^2}{\partial x^2}(x\psi)\,dx - \frac{\hbar}{2mi}\int_{-\infty}^{\infty} x\,\psi^*\frac{\partial^2\psi}{\partial x^2}\,dx \overset{?}{=} \frac{\langle p\rangle}{m}$$

$$\frac{d}{dx}\langle x\rangle = \frac{\hbar}{2mi}\int_{-\infty}^{\infty}\psi^*\left(2\frac{\partial\psi}{\partial x} + x\frac{\partial^2\psi}{\partial x^2}\right) - x\psi^*\frac{\partial^2\psi}{\partial x^2}\,dx$$

$$= \frac{\hbar}{mi}\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}\,dx \overset{?}{=} \frac{\langle p\rangle}{m}$$

$$\langle p\rangle = \int_{-\infty}^{\infty}\hbar k\,P_k(k)\,dk = \int_{-\infty}^{\infty}|A(k)|^2\,\hbar k\,\frac{dk}{2\pi} \qquad P_k(k) = \frac{|A(k)|^2}{2\pi}$$

$$\psi(x) = \int_{-\infty}^{\infty}A(k)e^{ikx}\frac{dk}{2\pi}$$

$$\frac{\partial}{\partial x}\psi(x) = \int_{-\infty}^{\infty}A(k)\,i\hbar\,e^{ikx}\frac{dk}{2\pi}$$

$$\psi^*(x) = \int_{-\infty}^{\infty}A^*(k)e^{ikx}\frac{dk}{2\pi}$$

$$\int_{-\infty}^{\infty}\psi^*\frac{d\psi}{dx}\,dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}A^*(k)e^{-ikx}\frac{dk}{2\pi}\right]$$

$$\times\left[\int_{-\infty}^{\infty}A(k)ike^{ikx}\frac{dk}{2\pi}\right]dx$$

$$= \int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dk'\int_{-\infty}^{\infty}dx\left\{A^*(k)A(k')\,ik'\right.$$

$$= \int_{-\infty}^{\infty}e^{i(k'-k)x}\,dx = 2\pi\delta(k'-k)$$

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\left\{A^*(k)A(k')\,ik'\,2\pi\delta(k'-k)\right\} = \int_{-\infty}^{\infty}|A(k)|^2 ik\,\frac{dk}{2\pi}$$

$$\langle p\rangle = \frac{\hbar}{mi}\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}\,dx = \frac{\hbar}{mi}\int_{-\infty}^{\infty}|A(k)|^2\, ik\,\frac{dk}{2\pi}$$

$$= \frac{\int_{-\infty}^{\infty}|A(k)|^2\hbar k\,\frac{dk}{2\pi}}{m} = \frac{\langle p\rangle}{m}$$

$$\langle p\rangle = \int_{-\infty}^{\infty}\psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\psi\right)dx$$

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} = i\hbar\frac{\partial}{\partial x}$$

what is an operator? / macro

macro pattern... replacement

$$\langle \hat{p}\rangle = \int_{-\infty}^{\infty}|A(k)|^2\hbar k\frac{dk}{2\pi} \Rightarrow \hat{p} = \hbar k$$

thinking in k space or real space?

$$= \int_{-\infty}^{\infty}\psi^*\hat{p}\psi\,dx$$

pattern replacement

$$\langle\hat{p}\rangle = \int_{-\infty}^{\infty}A^*(k)\,\hat{p}\,A(k)\frac{dk}{2\pi}$$

$$\hat{p}\,e^{ikx} = i\hbar\frac{\partial}{\partial x}e^{ikx}$$

$$= \hbar k e^{ikx}$$

linear map in a matrix?

higher level abstract construct?

k-space thing... what does that even mean?

Start assembling a list of operators

$\cancel{\hat{p} = -i\hbar \frac{\partial}{\partial x}}$

operators / x	K
$\hat{p} = i\hbar\frac{\partial}{\partial x}$	$\hbar K$
$\hat{X} = X$	$i\frac{\partial}{\partial K}$
$i\hbar\frac{\partial}{\partial t}$ $\quad \hat{E}$	
$\hat{V}$ $\quad V(x)$	Fourier transform of V

$$i\hbar\frac{\partial}{\partial t}e^{-i\omega t} = \hbar\omega e^{-i\omega t}$$

	x	K
$\hat{KE} = \hat{T}$	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$	$\frac{(\hbar k)^2}{2m}$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$$ ⟸ hamiltonian

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi$$

$$\hat{E}\psi = \hat{T}\psi + \hat{V}\psi$$

$$= H\psi$$

look up ehrenfest's theorem

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{d}{dt}\int_{-\infty}^{\infty}\psi^*\hat{Q}\psi\,dx = \int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial t}\hat{Q}\psi\,dx$$

$$+\int_{-\infty}^{\infty}\psi^*\frac{\partial\hat{Q}}{\partial t}\psi\,dx$$

$$\frac{\partial\psi}{\partial t} = \frac{\hat{H}\psi}{i\hbar}$$

$$\frac{\partial\psi^*}{\partial t} = \frac{\hat{H}^*\psi^*}{-i\hbar}$$

$$\frac{d}{dt}\langle\hat{Q}\rangle = \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle + \int_{-\infty}^{\infty}\psi^*\hat{Q}\frac{\partial\psi}{\partial t}\,dx \quad \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

$$+\int_{-\infty}^{\infty}\left(\frac{\hat{H}^*\psi^*}{-i\hbar}\right)\hat{Q}\psi\,dx$$

$$+\int_{-\infty}^{\infty}\psi^*\hat{Q}\frac{\hat{H}\psi}{i\hbar}\,dx$$

$$\frac{d}{dt}\langle\hat{Q}\rangle = \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle - \frac{1}{i\hbar}\int\hat{H}^*\psi^*\hat{Q}\psi\,dx +$$

$$\frac{1}{i\hbar}\int\psi^*\hat{Q}\hat{H}\psi\,dx$$

$$\int(\hat{H}\psi)^*\hat{Q}\psi\,dx$$

$$= \int\psi^*(\hat{H}\hat{Q}\psi)\,dx$$

$$\hat{Q}\hat{H} - \hat{H}\hat{Q} = [\hat{Q},\hat{H}]$$

$$\frac{d}{dt}\langle Q\rangle = \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle - \frac{1}{i\hbar}\int\psi^*\hat{H}\hat{Q}\psi$$

$$+\psi^*\hat{Q}\hat{H}\psi\,dx$$

$$= \left\langle\frac{\partial\hat{Q}^*}{\partial t}\right\rangle + \frac{1}{i\hbar}\int\psi^*(\hat{Q}\hat{H} - \hat{H}\hat{Q})\,dx$$

$$\boxed{\frac{d}{dt}\langle\hat{Q}\rangle = \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[\hat{Q},\hat{H}]\rangle}$$