11/13 6.728 Lecture 28

decay? Scattering

From last ~~time~~ time

$E \to E - q\Phi$

$\hat{p} \to \hat{p} - q\vec{A}$

$$E\psi = \frac{\hat{p}\cdot\hat{p}}{2m}\psi \Rightarrow (E - q\Phi)\psi = \frac{(\hat{p} - q\vec{A})(\hat{p} - q\vec{A})}{2m}$$

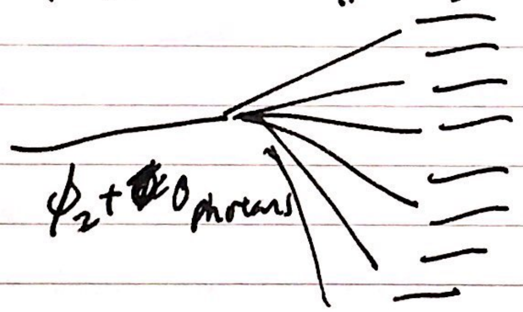
model for particle + E+M field

$$\hat{H} = \hat{H}_{EM} + \hat{H}_{particle} + \hat{H}_{int}$$

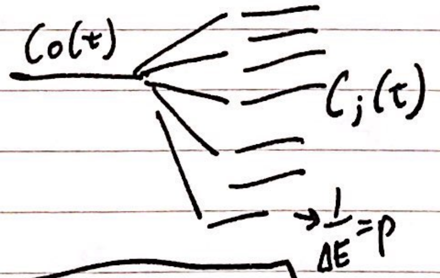
$$\hat{H}_{EM} = \int \tfrac{1}{2}\epsilon_0 \hat{\vec{E}}\cdot\hat{\vec{E}} + \tfrac{1}{2}\mu_0 \hat{\vec{H}}\cdot\hat{\vec{H}}\, d^3r = \sum_k \sum_\sigma \hbar\omega_k\left(\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma} + \tfrac{1}{2}\right)$$

$$\hat{H}_{particle} = \frac{\hat{p}\cdot\hat{p}}{2m} + q\Phi\left(+ q\frac{\vec{A}\cdot\vec{A}}{2m}\right)$$

$$\hat{H}_{int} = -\frac{q}{m}(\hat{\vec{A}}\cdot\hat{\vec{p}})$$



Dirac (1927) model



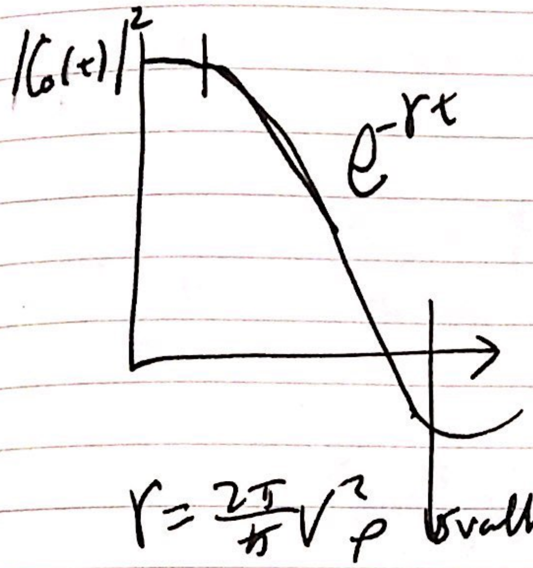
$$i\hbar\frac{d}{dt}C_0(t) = E_0 C_0(t) + \sum_j V_{0j} C_j(t)$$

$$i\hbar\frac{d}{dt}C_j(t) = E_j C_j(t) + V_{j0} C_0(t)$$

$$C_j(t) = \int_0^t \frac{|V_{0j}|^2}{i\hbar} e^{-\frac{[E\cdot(t-t')]}{i\hbar}} C_0(t')\, dt'$$

$$\boxed{\gamma = \frac{2\pi}{\hbar} V^2 \rho}$$

What model would actually give you exponential decay?

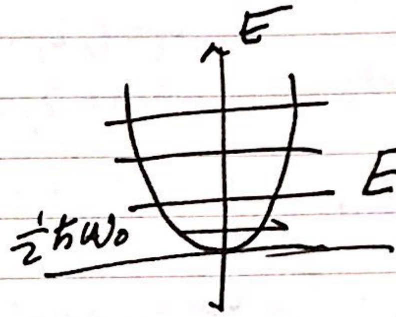


$$\gamma = \frac{2\pi}{\hbar}\left[|V(E)|^2 \rho(E)\right]_{E=E_0}$$

$\gamma = \frac{2\pi}{\hbar} V^2 \rho$ usually useless in current form.

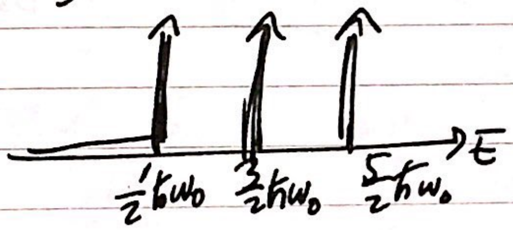
Density of States

$\rho(E) = g(E)$ = density of states



$E = \hbar\omega_0(n+\frac{1}{2})$

$P(E) =$

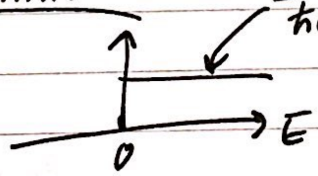


$$\int_{-\infty}^{\infty} \rho(E)Q(E)\,dE = \int_{-\infty}^{\infty} \sum_n \delta(E-\overbrace{\hbar\omega_0(n+\tfrac{1}{2})}^{E_n})\,Q(E)\,dE$$

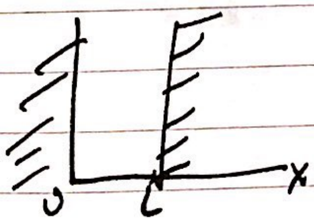
$$= \sum_n Q(E_n) = \sum_n Q_n$$

$$\sum_n Q_n = \int_{-\infty}^{\infty} Q(E)\rho(E)\,dE$$

Continuum limit



$$n = \sqrt{\frac{2mL^2}{\hbar^2\pi^2}E}$$



$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

$$E_{n+1} - E_n = \frac{\hbar^2\pi^2}{2mL^2}(n^2+2n+1) - \frac{\hbar^2\pi^2}{2mL^2}n^2 = \frac{\hbar^2\pi^2}{2mL^2}(2n+1)$$

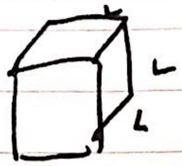
$$\rho(E) \to \begin{cases} 0 & E<0 \\ \sqrt{\dfrac{mL^2}{2\pi^2\hbar^2 E}} & E>0 \end{cases}$$

$$\rho_{n+1/2} = \frac{1}{E_{n+1}-E_n} = \frac{1}{\frac{\hbar^2\pi^2}{2mL^2}(2n+1)} = \frac{mL^2}{\hbar^2\pi^2 n}$$

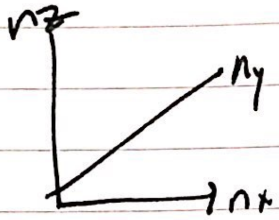
$$E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2$$

$$dE \quad \frac{\hbar^2\pi^2}{mL^2} n\,dn \qquad \rho = \frac{dn}{dE} = \frac{1}{\frac{\hbar^2\pi^2 n}{mL^2}}$$

Square well in 3D



$$E_{n_x n_y n_z} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2+n_y^2+n_z^2\right)$$



$n_x = 1, 2, \dots$

$n_y = 1, 2, \dots$

$n_z = 1, 2, \dots$

$\rho_{xy}(E) = (\rho_x + \rho_y)(E)$

$$n = \sqrt{n_1^2 + n_2^2 + n_3^2} \qquad E = \frac{\hbar^2\pi^2 n^2}{2mL^2} = n^2 E_1$$

$$N(n) = \frac{1}{8}\left(\frac{4\pi}{3} n^3\right)$$

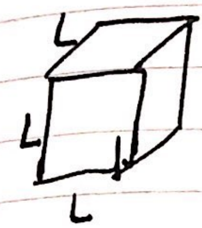
$$N(E) = \frac{1}{8}\,\frac{4\pi}{3}\left(\frac{2mL^2}{\hbar^2\pi^2} E\right)^3$$

$$\rho(E) = \frac{d}{dE}N(E) = \frac{1}{8}\,\frac{4\pi}{3}\left(\frac{2mL^2}{\hbar^2\pi^2}\right)^3 3E^2$$

$\Leftarrow$ volume of sphere within positive octant

$$\rho(E) = \begin{cases} 0 & E < 0 \\ \frac{\pi}{4}\left[\frac{2mL^2}{\hbar^2\pi^2}\right]^{3/2} E^{1/2} & E > 0 \end{cases}$$

Application: Metal at T=0



$$N = \sum_{\substack{n_x n_y n_z \\ \sigma}} 1 = \int_{-\infty}^{\infty} f(\epsilon) g(\epsilon) d\epsilon$$

$$= \int_0^{E_F} 1 \cdot g_s \cdot \frac{\pi}{4}\left[\frac{2mL^2}{\hbar^2\pi^2}\right]^{3/2} \epsilon^{1/2} d\epsilon \qquad f(\epsilon) = \begin{cases} 1 & \epsilon < E_F \\ 0 & \epsilon > E_F \end{cases}$$

spin degeneracy

$$\int_0^{E_F} \epsilon^{1/2} d\epsilon = \frac{2}{3} E_F^{3/2}$$

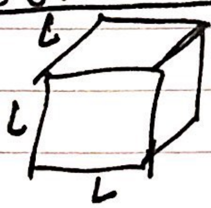
$$n_e = \frac{N}{L^3} = \frac{\pi}{6} g_s \left[\frac{2mE_F}{\hbar^2\pi^2}\right]^{3/2}$$

Why metals have conductivity at low temp

	n_e	E_F(eV)	V_F	cm/sec
Li	4.7×10^{22}/cm³		4.75	1.29×10^8
Be	24.7		14.4	2.23 "
Na	2.65		3.24	1.06 "
Al	18.1		11.66	2.02 "

electrons at Fermi surface have sizable amount of energy actually going quite fast

Efm field (free space) density of states

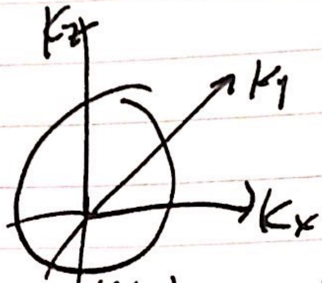


$e^{i\vec{k}\cdot\vec{r}}$ to be periodic $e^{i\vec{k}\cdot\vec{r}} = e^{i\vec{k}\cdot(\vec{r} + \hat{x} L)}$

$$\vec{k}_{n_x n_y n_z} = \hat{1}_x \frac{2\pi n_x}{L} + \hat{1}_y \frac{2\pi n_y}{L} + \hat{1}_z \frac{2\pi n_z}{L}$$

$$\boxed{\sum_{\vec{k}} Q(\vec{k}) \to \int Q(\vec{k}) \frac{d^3k}{(2\pi/L)^3} = L^3 \int Q(\vec{k}) \frac{d^3\vec{k}}{(2\pi)^3}}$$

Very important for stat mech



$$E = \hbar c k = \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$N(k) = g_\sigma L^3 \int 1 \frac{d^3\vec{k}}{(2\pi)^3} \longleftarrow \sum_{\sigma \vec{k}} 1 = \sum_{k} g_\sigma = \sum_{\vec{k}} 2$$

$g_\sigma \leftrightarrow 2$

$$= \frac{g_\sigma L^3}{(2\pi)^3}\left(\frac{4\pi}{3}k^3\right)$$

$10,\!000,\!000,\!000$
$= 1,\!000,\!000,\!000$

$$E = \hbar \omega_k = \hbar c k$$

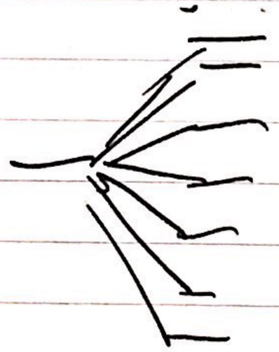
$$N(E) = \frac{g_\sigma L^3}{6\pi^2}\left(\frac{E}{\hbar c}\right)^3$$

$$\frac{d}{dE}N(E) = \frac{g_\sigma L^3}{6\pi^2}\frac{1}{(\hbar c)^3}3E^2$$

$$\rho(E) = \begin{cases} 0 & E<0 \\ \frac{L^3 E^2}{\pi^2 \hbar^3 c^3} & E>0 \end{cases}$$

<u>Radiative Decay</u>

$\phi_i(0)$
$+$
0 photons



$\phi_f(\vec{r}) + 1$ photon

$$\hat{H}_{EM} = \sum_{k\sigma} \hbar\omega_k \left(\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma} + \frac{1}{2}\right)$$

$$\Psi_i = \phi_i(\vec{r})|\Phi_0\rangle \quad \Phi_0 = |\Phi_0\rangle = \prod_{k\sigma}|n_{k\sigma} = 0\rangle$$

$$\Psi_f = \phi_f(\vec{r})\left(\hat{a}^\dagger_{k\sigma}|\Phi_0\rangle\right) = \phi_0(P_{k_1\sigma_1})\,\phi_0(P_{k_2\sigma_2})\ldots$$

Golden Rule

$$\gamma = \frac{2\pi}{\hbar} \left| \langle \Psi_f | \hat{H}_{int} | \Psi_i \rangle \right|^2 \rho \quad \left(\frac{L^3 \epsilon^2}{\pi^2 \hbar^3 c^3} \right)_{\epsilon = \hbar\omega = (E_f - E_i)}$$

$$M_{fi} = \langle \Psi_f | \hat{H}_{int} | \Psi_i \rangle = \left\langle \phi_f(\vec{r})\, \hat{a}^+_{k\sigma} | \Phi_0 \right\rangle \left| \frac{-q}{m} \hat{\vec{A}}(\vec{r}) \hat{\vec{p}} \right| \phi_i(r) | \Phi_0 \rangle$$

$$= \left\langle \phi_f \hat{a}^+_{k\sigma} \Phi_0 \right| -\frac{q}{m} \sum_{k\sigma} \hat{i}_\sigma \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k L^3}} \left(\frac{\hat{a}_{k\sigma} e^{i \vec{k}\cdot\vec{r}} - \hat{a}^+_{k\sigma} e^{-i\vec{k}\cdot\vec{r}}}{i} \right) \cdot \hat{\vec{p}} \left| \phi_i \Phi_0 \right\rangle$$

$$\left\langle \phi_f \right| -\frac{q}{m} \hat{i}_\sigma \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k L^3}} \left(-\frac{1}{i} \right) e^{-i\vec{k}\cdot\vec{r}} \cdot \hat{\vec{p}} \left| \phi_i \right\rangle$$

$$= -\frac{q}{m} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k L^3}} \langle \phi_f | \underbrace{e^{-i\vec{k}\cdot\vec{r}}}_{1 \text{ in dipole approx.}} \hat{i}_\sigma \cdot \hat{\vec{p}} | \phi_i \rangle$$

Next time → complete this → stat mech