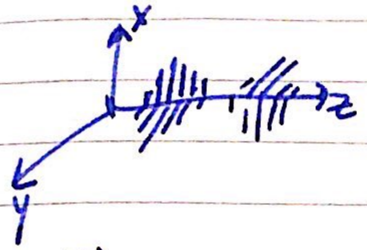
Lecture: Quantization of E+M fields, free space 26 11/6



$$\frac{d}{dt}e(t) = -i\frac{k}{\epsilon_0}h(t)$$

$$\frac{d}{dt}h(t) = -i\frac{k}{\mu_0}e(t)$$

$$\vec{E}(\vec{r},t) = \hat{i}_x e(t) e^{ikz}$$
$$\vec{H}(\vec{r},t) = \hat{i}_y h(t) e^{ikz}$$

$$\frac{d^2}{dt^2}e(t) = \frac{-ik}{\epsilon_0}\frac{d}{dt}h(t) = \frac{-k^2}{\epsilon_0\mu_0}e(t) = -\omega_0^2 e(t)$$

$$\hookrightarrow e(t) = Ae^{-i\omega_0 t} + \underbrace{\cancel{Be^{i\omega_0 t}}}_{0}$$

$e(t) \rightarrow \hat{e} = \text{const}\,\hat{a}$

$e^\dagger(t) \rightarrow \hat{e}^\dagger = \text{const}\,\hat{a}^\dagger$

Want to find const.

$$\frac{d}{dt}e(t) = -i\omega_0 e(t) \longrightarrow \frac{d}{dt}\langle \hat{a}\rangle = -i\omega_0\langle\hat{a}\rangle$$

$$\frac{k}{\epsilon_0}h(t) = \omega_0 e(t)$$

From 1st Quantization

$$\int \frac{1}{2}\epsilon_0\left(\vec{E}^*\vec{E} + \vec{E}\cdot\vec{E}^*\right) + \frac{1}{2}\mu_0\left(\vec{H}^*\vec{H} + \vec{H}\cdot\vec{H}^*\right)d^3\vec{r} = \hbar\omega_0$$

$$\hookrightarrow \int \frac{1}{2}\epsilon_0|\text{const}|^2\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{1}{2}\mu_0|\text{const}|^2\left(a^\dagger\hat{a} + aa^\dagger\right)d^3\vec{r} = \hat{H}$$

Also from classical

$$w = \int \frac{1}{2}\epsilon_0\vec{E}\cdot\vec{E} + \frac{1}{2}\mu_0\vec{H}\cdot\vec{H}\,d^3\vec{r}$$

$$\hookrightarrow \hat{H} = \int \frac{1}{2}\epsilon_0\left(\text{const}\,\hat{a}e^{ikz} + \text{const}^*\hat{a}^\dagger e^{-ikz}\right)^2 + (\text{equation for } \hat{H})\,d^3\vec{r}$$

$$\Rightarrow \text{const} = \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}} \qquad \vec{E}(\vec{r}) = \hat{i}_x\hat{e}e^{ikz} + \hat{i}_x\hat{e}^\dagger e^{-ikz}$$

$$\vec{E}(\vec{r}) = \hat{i}_x\sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}}\left(\hat{a}e^{ikz} + \hat{a}^\dagger e^{-ikz}\right) \qquad \vec{H}(r) = \hat{i}_y\sqrt{\frac{\hbar\omega_0}{2\mu_0 L^3}}\left(\hat{a}e^{ikz} + \hat{a}^\dagger e^{-ikz}\right)$$

Multi-mode case

$$\vec{E}(\vec{r}) = \sum_{k}\sum_{\sigma}\hat{i}_\sigma\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}}\left(\hat{a}_{k\sigma}e^{i\vec{k}\cdot\vec{r}} + \hat{a}^\dagger_{k\sigma}e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\vec{H}(\vec{r}) = \sum_{k}\sum_{\sigma}\hat{i}_k\times\hat{i}_\sigma\sqrt{\frac{\hbar\omega_k}{2\mu_0 L^3}}\left(\hat{a}_{k\sigma}e^{i\vec{k}\cdot\vec{r}} + \hat{a}^\dagger_{k\sigma}e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\hat{H} = \int\left(\frac{1}{2}\epsilon_0\vec{E}(\vec{r})\cdot\vec{E}(\vec{r}) + \frac{1}{2}\mu_0\hat{\vec{H}}(\vec{r})\vec{H}(\vec{r})\right)d^3\vec{r}$$
$$= \sum_{k\sigma}\hbar\omega_k\left(\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma} + \frac{1}{2}\right)$$

$E(\vec{r},t) \longleftrightarrow \langle\hat{\vec{E}}(\vec{r})\rangle$ Use Ehrenfest's theorem can produce Maxwell's eq'ns.

$$\nabla\cdot\langle\hat{\vec{E}}(\vec{r})\rangle=0$$

$$\nabla\cdot\langle\hat{\vec{H}}(\vec{r})\rangle=0$$

$$\nabla\times\langle\hat{\vec{E}}(\vec{r})\rangle=-\frac{\partial}{\partial t}\mu_0\langle\hat{\vec{H}}(\vec{r})\rangle$$

$$\nabla\times\langle\hat{\vec{H}}(\vec{r})\rangle=\frac{\partial}{\partial t}\epsilon_0\langle\hat{\vec{E}}(\vec{r})\rangle$$

$$\hat{\vec{S}}(\vec{r})=\hat{\vec{E}}(\vec{r})\times\hat{\vec{H}}(\vec{r})$$

$$\frac{\partial}{\partial t}\frac{1}{2}\epsilon_0\langle\hat{\vec{E}}\cdot\hat{\vec{E}}\rangle+\nabla\cdot\langle\vec{S}(\vec{r})\rangle=0$$
$$+\frac{1}{2}\mu_0\langle\hat{H}\cdot\hat{H}\rangle$$

additional issues:
- vector & scalar potential, gauge
- (...) providers?

$\Rightarrow$ Golden Rule

Scalar + Vector Potentials

☆ THIS IS IMPORTANT

Φ = Scalar potential

$\vec{A}$ = Vector potential (not observable)

$$\vec{E}(\vec{r},t)=-\nabla\Phi(\vec{r},t)-\frac{\partial}{\partial t}\vec{A}(\vec{r},t)$$

$$\mu_0\vec{H}(\vec{r},t)=\nabla\times\vec{A}(\vec{r},t)$$

Initially introduced for elasticity.

allow you to separate longitude & transverse part

$$\nabla\cdot\epsilon_0\vec{E}(\vec{r},t)=\epsilon_0\left[-\nabla^2\Phi-\nabla\cdot\frac{\partial\vec{A}}{\partial t}\right]=\rho$$

$$-\nabla^2\Phi=\rho_{\epsilon_0}+\frac{\partial}{\partial t}(\nabla\cdot\vec{A})$$

relationship between scalar & vector potential & charge density.

$$\nabla\cdot\mu_0\vec{H}=\nabla\cdot(\nabla\times\vec{A})=0$$

$$\nabla\times\vec{E}=-\frac{\partial}{\partial t}\mu_0\vec{H}$$
$$\downarrow$$
$$\nabla\times\left[-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}\right]=-\frac{\partial}{\partial t}\nabla\times\vec{A}$$

$$-\frac{\partial}{\partial t}\nabla\times\vec{A}=-\frac{\partial}{\partial t}\nabla\times\vec{A}$$

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial}{\partial t}\epsilon_0\vec{E}$$

$$\frac{1}{\mu_0}\underbrace{\nabla\times\nabla\times\vec{A}}_{-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})}=\vec{J}+\frac{\partial}{\partial t}\epsilon\left(-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}\right)$$

$$\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla^2\vec{A}=\mu_0\vec{J}-\nabla\left[\frac{1}{c^2}\frac{\partial\Phi}{\partial t}+\nabla\cdot\vec{A}\right]$$

Gauge

Radiation: $\frac{1}{c^2}\frac{\partial}{\partial t}\Phi+\nabla\cdot\vec{A}=0$

$$\Rightarrow\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}-\nabla^2\vec{A}=\mu_0\vec{J}$$

$$*\quad\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi-\nabla^2\Phi=\rho_{\epsilon_0}\quad\leftarrow\text{dynamical eq'n}$$

"dynamical degrees of freedom"
only 3...?

Coulomb Gauge

$$\nabla\cdot\vec{A}=0\rightarrow-\nabla^2\Phi=\frac{\rho}{\epsilon_0}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}-\nabla^2\vec{A}=\mu_0\vec{J}-\nabla\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)$$

Function: remove longitudinal part.
Splits problem into a scalar problem
$\rightarrow$ static problem.

(Note problem 5 in Pset for intuition)

Coulomb Gauge

$\vec{E} = -\nabla \Phi - \frac{\partial}{\partial t}\vec{A}$ * Separation of longitudinal & transverse *

$\uparrow \vec{E}_L$ $\nwarrow \vec{E}_T$

$\mu_0 \vec{H} = \nabla \times \vec{A}$

$\nwarrow \vec{H}_T$

(until someone discovers magnetic monopoles. (...) ?)

our quantization was for the H_T only. Longitudinal: static field.

Can I make a laser with a longitudinal electric field?

Something about vacuums?

*(from pg before) A different kind of quantization. Some degenerate kind of quantization.

Interaction Hamiltonian

$$\hat{H} = \frac{\hat{P}\cdot\hat{P}}{2m} = \frac{-\hbar^2\nabla^2}{2m}$$

$$|\hat{p}|^2 = |i\hbar\nabla|^2 = -\hbar^2\nabla^2$$

$\to \vec{p}\cdot\vec{p}$

Classical version was known prior to Dirac-

$\vec{p}(t) \longrightarrow \vec{p}(t) - q\vec{A}(\vec{r},t)$

$E(t) \longrightarrow E(t) - q\Phi(\vec{r},t)$

} modify theory to include electric fields & magnetic fields.

Classical $E = \frac{\vec{p}(t)\vec{p}(t)}{2m}$

$\hookrightarrow (E - q\Phi) = \frac{(\vec{p}-q\vec{A})(\vec{p}-q\vec{A})}{2m}$

How do you connect quantum models w/ electric-magnetic field?

$\hat{P} \rightarrow \hat{P} - q\hat{A}$

$\hat{E} \rightarrow \hat{E} - q\Phi$

doesn't work for spin-3 half problem

Minimal coupling

$$\hat{H} = \frac{\hat{P}\cdot\hat{P}}{2m} \Rightarrow \hat{H} - q\Phi = \frac{(\hat{P}-q\hat{A})(\hat{P}-q\hat{A})}{2m}$$

No field

$\hat{E}\psi = \hat{H}\psi$

$$\hat{H} = \frac{(\hat{p}-q\hat{A})(\hat{p}-q\hat{A})}{2m} + q\Phi$$

$\leftarrow$ external field

$$E = \sqrt{(mc^2)^2 + c^2 \vec{p}\cdot\vec{p}}$$

$$E - q\Phi = \sqrt{(mc^2)^2 + c^2(\vec{p}-q\vec{A})(p-q\vec{A})}$$

where did Lorentz force come from?

$$\hat{H} = \frac{(\hat{\vec{p}}-q\vec{A})(\hat{\vec{p}}-q\vec{A})}{2m} + q\Phi$$

$$= \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m} - \left(\frac{q\vec{A}\hat{\vec{p}} + \hat{\vec{p}}q\hat{\vec{A}}}{2m}\right) + \frac{q^2\vec{A}\cdot\hat{\vec{A}}}{2m} + \underbrace{q\Phi}_{V(\vec{r})}$$

$$= \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m} + V(\vec{r}) - \frac{q}{2m}\left(\vec{A}\cdot\hat{\vec{p}} + \hat{\vec{p}}\cdot\vec{A}\right) + \underbrace{\frac{q^2}{2m}|A|^2}_{\text{Ponderomotive Potential}}$$

What would you need to get a big $|A|^2$?

$\Rightarrow$ Big magnetic field

$\left(-\frac{q}{m}\vec{A}\cdot\vec{p}\right)$ Coulomb Gauge

$$\hat{H} = \hat{H}_{EM} + \hat{H}_{particle} + H_{int}$$

$$\hat{H}_{EM} = \int \tfrac{1}{2}\epsilon_0 \hat{\vec{E}}(\vec{r})\hat{\vec{E}}(\vec{r}) + \tfrac{1}{2}\mu_0 \hat{\vec{H}}(\vec{r})\hat{\vec{H}}(\vec{r})\, d^3\vec{r} = \sum_{k\sigma} \hbar\omega_k\left(\hat{a}^{+}_{k\sigma}\hat{a}_{k\sigma} + \tfrac{1}{2}\right)$$

$$\hat{H}_{particle} = \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m} + V(\vec{r})$$

$\hat{H}_0$

$\hat{V}$

$$\hat{H}_{int} = \frac{-q}{2m}\left(\hat{A}(\vec{r})\cdot\hat{\vec{p}} + \hat{\vec{p}}\cdot\hat{A}(\vec{r})\right)$$

momentum operator talks to vector potential

No field

$$\hat{H} = \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m} + V(\vec{r})$$

E_2 — $\rightarrow$ ~~Haw~~ Stay in state forever.

E_1 —

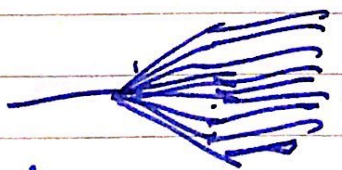
+ Field $\quad \hat{H}_0 + \hat{V}$

— $E_2 + \sum_{k\sigma}\frac{\hbar\omega_k}{2}$

— $E_1 + \sum_{k\sigma}\frac{\hbar\omega_k}{2}$

ground state + photon

— $E_1 + \hbar\omega_0 + \sum_{k\sigma}\frac{\hbar\omega_k}{2}$

$C_0(t)$  $C_j(t)$

$$i\hbar\frac{d}{dt}C_j(t) = E_jC_j(t) + i\langle\phi_j|\hat{V}|\phi_0\rangle C_0(t)$$

$$i\hbar\frac{d}{dt}C_0(t) = E_0C_0(t) + \sum_j\langle\phi_0|\hat{V}|\phi_j\rangle C_j(t)$$

$$\gamma = \frac{2\pi}{\hbar}|\langle\phi_0|\hat{V}|\phi_j\rangle|^2\rho$$

density of states

$E_j = E_0$