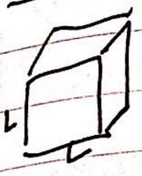
Lecture 6.728 25 11/4

Quantization of E&M field



$\vec{\nabla}\cdot\epsilon_0\vec{E}=0$

$\nabla\cdot\mu_0\vec{H}=0$

$\nabla\times\vec{E}=-\frac{\partial}{\partial t}\mu_0\vec{H}$

$\nabla\times\vec{H}=\frac{\partial}{\partial t}\epsilon_0\vec{E}$

resonator:

$\vec{E}(\vec{r},t)=\vec{u}(\vec{r})e(t)$

$\vec{H}(\vec{r},t)=\vec{v}(\vec{r})h(t)$

$\nabla\cdot\vec{u}(\vec{r})=0 \qquad \nabla\times\vec{u}(\vec{r})=k\vec{v}(r)$

$\nabla\cdot\vec{u}(\vec{r})=0 \qquad \nabla\times\vec{v}(\vec{r})=k\vec{u}(r)$

$\frac{d}{dt}e(t)=\frac{k}{\epsilon_0}h(t)$

$\frac{d}{dt}h(t)=\frac{-k}{\mu_0}e(t)$

$W=\int\frac{1}{2}\epsilon_0\vec{E}\cdot\vec{E}+\frac{1}{2}\mu_0\vec{H}\cdot\vec{H}d\vec{r}=\frac{1}{2}\epsilon_0L^3e^2(t)+\frac{1}{2}\mu_0L^3h^2(t)$

$\hat{H}=\frac{1}{2}\mu_0L^3\hat{h}^2+\frac{1}{2}\epsilon_0L^3e^2 \qquad \hat{h}=i\hbar A\frac{\partial}{\partial\rho}$

$=-\frac{1}{2}\mu_0L^3\hbar^3A^2\frac{d^2}{d\rho^2}\frac{1}{2}\epsilon_0L^3e^2$

$E\psi(e)=A\psi(e) \qquad \psi=e^{-\alpha e^2/2} \qquad \frac{d^2}{de^2}=[(\alpha e)^2-\alpha]\exp(-\alpha e^2/2)$

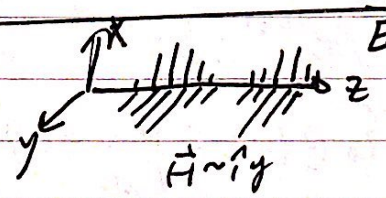
$\frac{d}{de}\psi=R(e)e^{-\alpha e^2/2}$

$\frac{\hbar\omega_0}{2}\exp\{\}=-\frac{1}{2}\mu_0L^3\hbar^2A^2[(\alpha e^2-\alpha]\exp\{\}+\frac{1}{2}\epsilon_0L^3e^2\exp\{\}$

$\frac{\hbar\omega_0}{2}=\frac{1}{2}\mu_0L^3\hbar^2A^2\alpha \quad \ldots \qquad A=\frac{\omega c}{L^3}$

$\frac{1}{2}\mu_0L^3A^2\alpha^2=\frac{1}{2}\epsilon_0L^3\ldots$

Free Space



Classical soln

$\vec{E}(\vec{r},t)=\text{const}\ \hat{i}_x\cos(kz-\omega_0t)$

$\vec{H}(\vec{r},t)=\text{const.}\ \hat{i}_y\cos(kz-\omega_0t)$

$W=\int\frac{1}{2}\epsilon_0\vec{E}\vec{E}+\frac{1}{2}\mu_0\vec{H}\vec{H}d^3r=\hbar\omega_0$

$E(\vec{r},t)=\sqrt{\frac{2\hbar\omega_0}{\epsilon_0L^3}}\hat{i}_x\cos(kz-\omega t)$

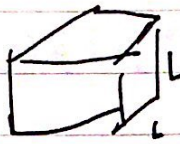
$\vec{H}(\vec{r},t)=\sqrt{\frac{2\hbar\omega_0}{\mu_0L^3}}\hat{i}_y\cos(kz-\omega_0t)$

1st Quantization

$$\vec{E}(r,t)\Big|_{1st} = \hat{i}_x \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}}\, e^{ikz} e^{-i\omega_0 t}$$

$$\int \frac{1}{2}\epsilon_0\left(\vec{E}^*\cdot\vec{E} + \vec{E}\cdot\vec{E}^*\right) + \frac{1}{2}\mu_0\left(\vec{H}^*\cdot\vec{H} + \vec{H}\cdot\vec{H}^*\right) d^3\vec{r}\,\Big|_{1st} = \hbar\omega_0$$

$$\vec{H}(r,t)\Big|_{1st} = \hat{i}_y \sqrt{\frac{\hbar\omega_0}{2\mu_0 L^3}}\, e^{ikz} e^{-i\omega_0 t}$$



L

$$\frac{\partial}{\partial t}\begin{bmatrix}\epsilon_0 \vec{E} \\ \mu_0 \vec{H}\end{bmatrix} = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0\end{bmatrix}\begin{bmatrix}\vec{E} \\ \vec{H}\end{bmatrix}$$

(no one has done ~~to~~ this... ?)

Start with 1st Quant

$$\vec{E}(\vec{r},t)\Big|_{1st} = \hat{i}_x\, e(t)\, e^{ikz}$$

$$\vec{H}(\vec{r},t)\Big|_{1st} = \hat{i}_y\, h(t)\, e^{ikz}$$

take advantage that frequencies have to be positive

$$\frac{d}{dt}e(t) = -\frac{ik}{\epsilon_0}h(t) = i\omega_0 e(t) \longrightarrow e(t) = \text{const}\; e^{-i\omega_0 t}$$

$$\frac{k}{\epsilon_0}h(t) = \omega_0 e(t)$$

$$\frac{d}{dt}h(t) = -i\frac{k}{\mu_0}e(t)$$

$$\frac{d}{dt}e(t) = -i\omega_0 e(t)$$

$$\Longrightarrow \frac{d^2}{dt^2}e(t) = \frac{-ik}{\epsilon_0}\frac{dh(t)}{dt} = \frac{-k^2}{\epsilon_0\mu_0}e(t)$$

$$= -\omega_0^2 e(t)$$

$$\longrightarrow e(t) = Ae^{-i\omega_0 t} + Be^{i\omega_0 t}$$

$$h(t) = \frac{\omega_0}{k}\epsilon_0 e(t) = c\epsilon_0 e(t) = \frac{1}{\sqrt{\epsilon_0\mu_0}}\epsilon_0 e(t) = \sqrt{\frac{\epsilon_0}{\mu_0}}\, e(t)$$

$$\boxed{\frac{d}{dt}e(t) = -i\omega_0 e(t)}$$

$\frac{d}{dt}\langle\hat{Q}\rangle = -i\omega_0\langle\hat{Q}\rangle$? $\quad \hat{H} = \hbar\omega_0(\hat{a}^+\hat{a}+\frac{1}{2})$

$\frac{d}{dt}\langle\hat{a}\rangle = -i\omega_0\langle\hat{a}\rangle \quad \frac{d}{dt}\langle\hat{a}\rangle = \frac{1}{i\hbar}\langle[\hat{a},\hat{H}]\rangle = -i\omega_0\langle\hat{a}\rangle$

$\hat{e} = \text{const}\ \hat{a} \quad e(t)\leftrightarrow\hat{e} \quad \frac{d}{dt}e^+(t) = +i\omega_0 e^+(t)$

$\frac{d}{dt}\langle\hat{a}^+\rangle = +i\omega_0\langle\hat{a}^+\rangle \quad e^+(t)\leftrightarrow\hat{e}^+$

Adjoint:

$\int(\hat{B}\psi)^*\hat{Q}\psi\,dx = \int\psi^*\hat{B}^+\hat{Q}\psi\,dx \Rightarrow \hat{B}^+ = \text{adjoint of } \hat{B}$

$\int(\hat{H}\psi)^*\hat{Q}\psi\,dx = \int\psi^*\hat{H}\hat{Q}\psi\,dx \Rightarrow \hat{H}^+ = \hat{H} \Rightarrow$ self adjoint.

$$\left[\int\psi^*\hat{B}\psi\,dx\right]^* = \int(\hat{B}\psi)^*\psi\,dx = \int\psi^*\hat{B}^+\psi\,dx$$

$$= \langle\hat{B}^*\rangle \quad = \quad \langle\hat{B}^+\rangle$$

$$w = \int\frac{1}{2}\epsilon_0(\vec{E}^*\vec{E} + \vec{E}\cdot\vec{E}^*) + \frac{1}{2}\mu_0(\quad)d^3\vec{r}$$

$$\hookrightarrow \hat{H} = \int\frac{1}{2}\epsilon_0\left(\underbrace{\hat{\vec{E}}^+(\vec{r})\hat{\vec{E}}(\vec{r})}_{1st} + \hat{\vec{E}}(\vec{r})\hat{\vec{E}}^+(\vec{r})\right) + \frac{1}{2}\mu_0(\ldots)d_i ?$$

$$= \frac{1}{2}\epsilon_0 L^3|\text{const}|^2(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$$

$$+\frac{1}{2}\mu_0 L^3|\text{const}|^2(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) = \hbar\omega_0(\hat{a}^+\hat{a}+\frac{1}{2}) = \frac{1}{2}\hbar\omega_0 \quad \hat{a}^+\hat{a} + \hat{a}\hat{a}^+$$

$$\frac{\partial}{\partial t}\begin{vmatrix}\epsilon_0\vec{E}\\ \mu_0\vec{H}\end{vmatrix} = \begin{vmatrix}0 & \nabla\times\\ -\nabla\times & 0\end{vmatrix}\begin{vmatrix}\vec{E}\\ \vec{H}\end{vmatrix}$$

$$\text{Const} = \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}} \quad \text{const} = \sqrt{\frac{\hbar\omega_0}{2\mu_0 L^3}}$$

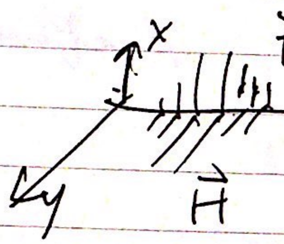
$$\hat{\vec{E}}(\vec r)\Big|_{1st} = \hat{i}_x \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}}\, e^{ikz}\,\hat a \qquad \vec{E}^{+}(\vec r)\Big|_{1st} = \hat{i}_x \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}}\, e^{ikz}\,\hat a^{+}$$

$$\hat H = \int \frac{1}{2}\epsilon_0\left(\vec E^{+}\vec E + \vec E\cdot\vec E^{+}\right) + \frac{1}{2}\mu_0\left(\vec H^{+}\vec H + \hat{\vec H}\hat{\vec H}^{+}\right) d^3 r$$

$$= \hbar\omega_0\left(\hat a^{+}\hat a + \frac{1}{2}\right) \rightarrow = \underbrace{\frac{\hbar\omega_0}{2}\left(-\frac{d^2}{dy^2} + y^2\right)}_{\text{configuration space}}$$

$$\vec E(\vec r, t)\Big|_{1st} \longleftrightarrow \left\langle \hat{\vec E}(\vec r)\Big|_{1st}\right\rangle$$

Multimode problem

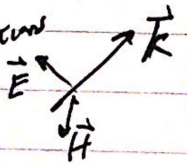


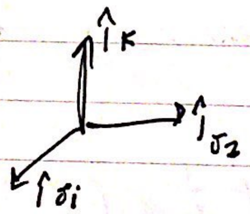
Resonator: $\hbar\omega_0\left(\hat a^{+}\hat a + \frac{1}{2}\right)$

$$\rightarrow \sum_j \hbar\omega_j\left(\hat a_j^{+}\hat a_j + \frac{1}{2}\right)$$

$$\hbar\omega_0\left(\hat a^{+}\hat a + \frac{1}{2}\right) \rightarrow \sum_k \sum_{\sigma=1}^{2} \hbar\omega_k\left(\hat a^{+}_{k\sigma}\hat a_{k\sigma} + \frac{1}{2}\right)$$

2 polarizations





$$\hat{\vec E}(\vec r) = \hat i_x e^{ikz}\sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}}\,\hat a \longrightarrow \hat{\vec E}(\vec r)\Big|_{1st} = \sum_{k,\sigma} \hat i_\sigma \sqrt{\frac{\hbar\omega_{|k|}}{2\epsilon_0 L^3}}\,\hat a_{k,\sigma}\, e^{i\vec k\cdot\vec r}$$

" most people don't reverse operators to 1st quantization even though it is particularly useful "

$$\hat{\vec E}(\vec r)\Big|_{classical} = \hat{\vec E}(\vec r) = \hat{\vec E}(\vec r)\Big|_{1st} + \hat{\vec E}^{+}(\vec r)\Big|_{1st}$$

$$\hat H = \int \frac{1}{2}\epsilon_0\, \hat{\vec E}(\vec r)\cdot\hat{\vec E}(\vec r) + \frac{1}{2}\mu_0\, \vec H(\vec r)\hat H(\vec r)\, d^2 r$$

$$= \hat i_x\, \text{const}\, e^{ikz}\hat a + \hat i_x\, \text{const}\, e^{-ikz}\hat a^{+}$$

$$\rightarrow \int \frac{1}{2}\epsilon_0 |const|^2 \left(e^{2ikz}\hat{a}\hat{a}^+ + (\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) + e^{-2ikz}(\hat{a}^+)^2 \right) d\vec{r}^{\,3}$$

$$+ (mag)$$

$$= \frac{1}{2}\epsilon_0 L^3 |const|^2 (\hat{a}^+ a + a a^+) + \frac{1}{2}\mu_0 L^3 (const)(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$$

$$\hat{\vec{E}}(\vec{r}) = \hat{i}_x \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 L^3}} \left(\hat{a}\, e^{ikz} + \hat{a}^+ e^{-ikz} \right)$$

$$\longrightarrow \sum_{k\sigma} \hat{i}_\sigma \sqrt{\frac{\hbar\,\omega_{|\vec{k}|}}{2\epsilon_0 L^3}} \left(\hat{a}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}} + \hat{a}^+_{k\sigma} e^{-i\vec{k}\cdot\vec{r}} \right)$$

$$\hat{H}(\vec{r}) = \sum_{\vec{k},\sigma} (\hat{i}_k \times \hat{i}_\sigma) \sqrt{\frac{\hbar\,\omega_{|\vec{k}|}}{2\mu_0 L^3}} \left(\quad '' \quad \right)$$

$\vec{E}(\vec{r},t)$ classical $\leftrightarrow \langle \vec{E}(\vec{r}) \rangle$

next time we will connect to matter