$$E = \frac{1}{2} L I^2(t) + \frac{1}{2} C V^2(t)$$

$$\hat{E}\Psi(V,t) = \frac{1}{2} L \hat{I}^2 \Psi(V,t) + \frac{1}{2} C V^2 \Psi(V,t)$$

$$i\hbar \frac{\partial}{\partial t}\psi(V,t) = -\frac{\hbar \omega_0^2}{2C^2}\frac{\partial^2}{\partial V^2}\psi(V,t) + \frac{1}{2} C V^2 \psi(V,t)$$

WKB unlikely on final

How to make quantum world

- guess $\hat{H}$
- work with analog problem to get form of $\hat{H}$
- work from classical Lagrangian (see notes)

$\Longrightarrow$ can verify using Ehrenfest.

Jamison: We're moving into 8.06.

Q: How do we know that it works?

New systems — predictive capabilities.

WKB - Jeffries, Wenzel, Kramers, Brill (...?)

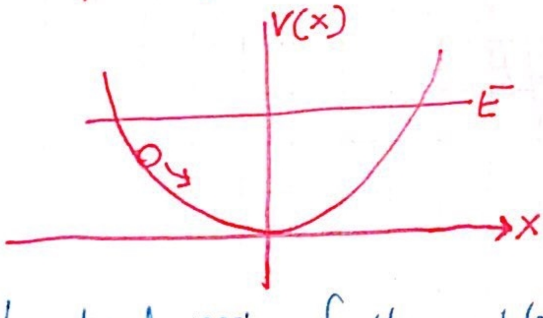
Think about the classical problem:

$$E = \frac{p^2(t)}{2m} + V(x(t)) \qquad \frac{d}{dt}x(t) = \frac{p(t)}{m} \qquad \frac{d}{dt}p(t) = -\frac{d}{dx}V(x(t))$$

$\Rightarrow p = \pm\sqrt{2m(E - V(x))} \longrightarrow \hbar k(x)$

$\psi \sim e^{ik(x)x}$ ← doesn't work very well.

$\sim e^{i\int^x k(x')dx'}$ flux conservati



1) Normalized version of the problem

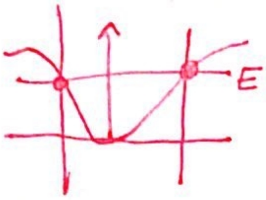
$$\epsilon\Psi(y) = -\frac{d^2}{dy^2}\psi(y) + v(y)\Psi(y)$$

$$\Psi(y) = \begin{cases} \dfrac{\sin\phi(y)}{\sqrt{\eta(y)}} \\ \dfrac{\cos\phi(y)}{\sqrt{\eta(y)}} \quad \dfrac{e^{\pm i\phi(y)}}{\sqrt{\eta(y)}} \end{cases} \qquad \eta(y) = \frac{d}{dy}\phi(y) \qquad \phi(y) = \int^y \eta(y')\,dy'$$

WKB transformation

$$\boxed{\epsilon = \eta^2(y) - \eta^{1/2}(y)\frac{d^2}{dy^2}\eta^{-1/2}(y) + v(y)}$$

$\Rightarrow$ non-linear differential eq'n.



The new nonlinear eq'n is actually really hard to solve.

However, you can get an approximate sol'n:

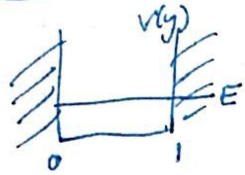
$$\epsilon \approx y'^2(y) + V(y) \qquad \eta(y) \approx \pm\sqrt{\epsilon - V(y)}$$

$$\psi(y) = \frac{\sin\phi(y)}{\eta^{1/2}} = \frac{\sin\left(\int^{y}\sqrt{\epsilon - V(y')}\,dy'\right)}{[\epsilon - V(y)]^{1/4}}$$

Near turning pts, •, WKB approximation not as accurate

Areas used: establishing boundary conditions for scattering.
Normalization of WKB sol'n

2) Application: establishing ϵ



$$\epsilon\psi(y) = -\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y)$$

$$\psi(y) = \sin(ky) \Rightarrow k = n\pi$$
$$\epsilon = n^2\pi^2$$

$$\psi(y) = \frac{\sin\phi}{\sqrt{\eta}} \qquad \eta = \frac{d\phi}{dy}$$

proposal make $0 = y_{min}$, $1 = y_{max}$ $\Rightarrow \int_{y_{min}}^{y_{max}} \eta(y)\,dy = \phi(y_{max}) - \phi(y_{min})$
$= n\pi$

JWKB

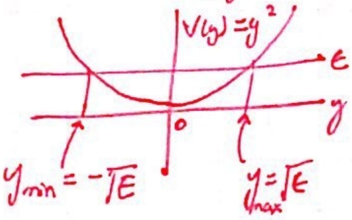
$$\eta = \sqrt{\epsilon - V(y)} \rightarrow \sqrt{\epsilon}$$

$$\int_0^1 \sqrt{\epsilon}\, d\epsilon = n\pi \qquad \sqrt{\epsilon} = n\pi \Rightarrow \epsilon_n = n^2\pi^2$$

numbering for square well

3) Try the Simple Harmonic Oscillator

$$\epsilon\psi = -\frac{d^2}{dy^2}\psi + y^2\psi$$



$$\int_{y_{min}}^{y_{max}} \eta(y)\,dy = \int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \sqrt{\epsilon - y^2}\,dy = \frac{\pi}{2}\epsilon$$
$$= \pi(n+1)$$

SHO numbering

For SHO, ground state at $n=0$

$\epsilon_n = 2n+2$ JWKB

$\epsilon_n = 2n+1$ exact

Revise phase Constraint

$$\int_{y_{min}}^{y_{max}} \eta(y)\,dy = (n+1)\pi - \frac{\pi}{4} - \frac{\pi}{4}$$

SHO · if left boundary in sd'n · if right boundary in sd'n

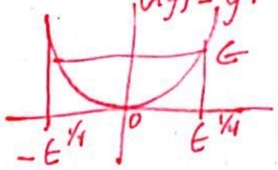
$$\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \sqrt{\epsilon - y^2}\,dy = \frac{\pi}{2}\epsilon = (n+1)\pi - \frac{\pi}{2}$$

$$= (n+\tfrac{1}{2})\pi$$

$$\epsilon_n = 2n+1$$

$$\epsilon\psi = -\frac{d^2}{dy^2}\psi + y^4\psi$$

JWKB : $\eta(y) = \sqrt{\epsilon - y^2}$



$$\int_{-\epsilon^{1/4}}^{\epsilon^{1/4}} \sqrt{\epsilon - y^4}\,dy = (n+\tfrac{1}{2})\pi = \frac{2\epsilon^{3/4}\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})}$$

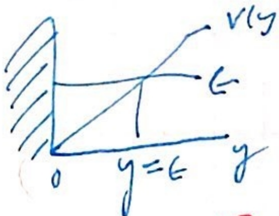
$$\epsilon_n = \left[\frac{\pi(n+\frac{1}{2})\Gamma(\frac{7}{4})}{2\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})}\right]^{4/3}$$

approximations

n	exact	JWKB
0	1.0604	0.8671
1	3.7997	3.7519
2	7.4557	7.4140
3	11.645	11.612
4	16.262	16.234

gets better as n gets larger

In his research, two-level system something high-order → he was able to use KWB to get an exact answer.



$$\epsilon_n = \left[\frac{3}{2}\left(n+\frac{3}{4}\right)\pi\right]^{2/3}$$

n	exact	WKB
0	2.3381	2.3203
1	4.0879	4.0818
2	5.5206	5.5172
3	6.7867	6.7845

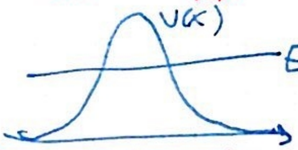
$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) \qquad k(x) = \pm\sqrt{\frac{2m(E-V(x))}{\hbar^2}}$$

$$\psi(x) = \frac{\sin(\phi(x))}{\sqrt{k(x)}}$$

$$E = -\frac{\hbar^2}{2m}k^{1/2}(x)\frac{d^2}{dx^2}k^{-1/2}(x) + V(x) + \frac{\hbar^2}{2m}k^2(x)$$

$$\int_{x_{min}}^{x_{max}} k(x)\,dx = (n+1)\pi - \frac{\pi}{4} - \frac{\pi}{4}$$

SHO numbering · soft left boundary · soft right boundary



$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x) \Rightarrow E = -\frac{\hbar^2}{2m}\alpha^{1/2}\frac{d^2\alpha^{-1/2}}{dx^2} + V(x)$$

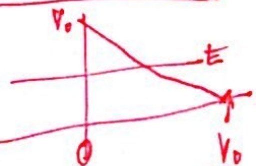
$$\psi(x) = \frac{e^{\pm\theta(x)}}{\sqrt{\alpha(x)}},\ \frac{\sinh\theta(x)}{\sqrt{\alpha(x)}},\ \frac{\cosh\theta(x)}{\sqrt{\alpha(x)}} \qquad -\frac{\hbar^2}{2m}\alpha^2$$

$\Rightarrow$ JWKB approx.

$$\alpha(x) = \pm\sqrt{\frac{2m(V(x)-E)}{\hbar^2}}$$

$$\Rightarrow T = |t|^2 \sim e^{-2G}$$

$$G = \int_{x_{min}}^{x_{max}} \alpha(x)\,dx$$



$$V(x) = V_0 - e\mathcal{E}x \qquad 0 < x < \frac{V_0}{e\mathcal{E}}$$

$\boxed{\text{tunneling through triangle barrier} \Rightarrow \text{MOSFET \& ... capacitor?}}$ ☆

$$G = \int_0^{\frac{V_0 E}{e\mathcal{E}}} \sqrt{\frac{2m(V_0 - e\mathcal{E}x - E)}{\hbar^2}}\,dx$$

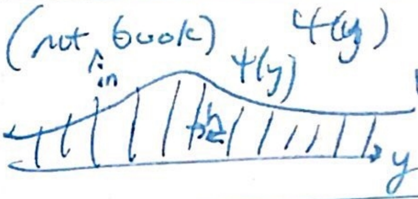
$$= \frac{2}{3}\sqrt{\frac{2m}{e\mathcal{E}}}\,(V_0 - E)^{3/2}/\hbar$$

WKB helps figure out how much exponential decay

$$T = e^{-2G}$$

see other page two ago for more notes

Numerical – Finite difference method (not book)



$y_n = nh$

$\psi(y_n) \to \psi_n$

$$\epsilon \psi(y) = -\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y)$$

$$\epsilon \psi(y_n) = -\left(\frac{d^2\psi}{dy^2}\right)_n + V(y_n)\psi(y_n)$$

$$\epsilon \psi_n = -\left(\frac{d^2\psi}{dy^2}\right) + V_n\psi_n$$

$$\boxed{\epsilon \psi_n = \frac{-\psi_{n+1} - 2\psi_n + \psi_{n-1}}{h^2} + V_n\psi_n + O(h^2)}$$

$$O(h^2) \leftarrow \frac{h^2}{12}\left(\frac{d^4\psi}{dy^4}\right)$$

$$\psi(y_n+1) = \psi(y_n) + h\left(\frac{d}{dy}\psi\right)_{y_n} + \frac{h^2}{2}\left(\frac{d^2\psi}{dy^2}\right)_n + \dots$$

$$\psi(y_n-1) = \psi(y_n) - h\left(\frac{d}{dy}\psi\right)_{y_n} + \frac{h^2}{2}\left(\frac{d^2\psi}{dy^2}\right)_n + \dots$$

$$\psi(y_n+1) + \psi(y_{n-1}) = 2\psi(y_n) + \frac{2h^2}{2}\left(\frac{d^2\psi}{dy^2}\right)_n + \frac{2h^4}{4!}\left(\frac{d^4\psi}{dy^2}\right)_n + \dots$$

$$\left(\frac{d^2\psi}{dy^2}\right)_n = \frac{\psi(y_n+1) - 2\psi(y_n) + \psi(y_{n-1})}{h^2} - \frac{h^2}{12}\left(\frac{d^4\psi}{dy^4}\right)_n \dots$$

$$\left(\frac{d^2\psi}{dy^2}\right)_n \approx \frac{\psi_{n+1} - 2\psi_n + \psi_n}{h^2}$$

$$\left(\frac{d^4\psi}{dy^4}\right)_n = \left(\frac{d}{dy^2}\left(\frac{d^2\psi}{dy^2}\right)\right)_n \approx \frac{\left(\frac{d^2\psi}{dy^2}\right)_{n+1} - 2\left(\frac{d^2\psi}{dy^2}\right)_n + \left(\frac{d^2\psi}{dy^2}\right)_{n-1}}{h^2}$$

$$\epsilon\,\frac{\psi_{n+1} + 10\psi_n + \psi_{n-1}}{12} = -\left(\frac{\psi_{n+1} - 2\psi_n + \psi_{n-1}}{h^2}\right) + \frac{1}{12}\left(V_{n+1}\psi_{n+1} + 10V_n\psi_n + V_{n-1}\psi_{n-1}\right)$$

Numerov Scheme