Lecture 12 Quantum Mechanical Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 x^2$$

$$\phi_n(x) = \left[\frac{m\omega_0}{\pi\hbar}\right]^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\frac{1}{2}\frac{m\omega_0}{\hbar}x^2} H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}\, x\right)$$

$$E_n = \hbar\omega_0 (n+\tfrac{1}{2})$$

$$\int_{-\infty}^{\infty} (\hat{a}\phi_n)^* \phi_n \, dx = \int_{-\infty}^{\infty} \phi_n^* (\hat{a}^+ \phi_n) dx$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}} x + \sqrt{\frac{\hbar}{m\omega_0}} \frac{d}{dy}\right) = \sqrt{\frac{m\omega_0}{2\hbar}} X + i\sqrt{\frac{1}{2m\hbar\omega_0}}\hat{p}$$

$$x = \sqrt{\frac{\hbar}{2m\omega_0}} (\hat{a} + \hat{a}^+)$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}} x - \sqrt{\frac{\hbar}{m\omega_0}} \frac{d}{dy}\right) = \sqrt{\frac{m\omega_0}{2\hbar}} x - i\sqrt{\frac{1}{2m\hbar\omega_0}}\hat{p}$$

$$\hat{p} = \sqrt{\frac{m\hbar\omega_0}{2}}\left(\frac{\hat{a} - \hat{a}^+}{i}\right)$$

Eigenvalue Relation

$$\hat{a}\phi_n = \sqrt{n}\,\phi_{n-1}$$
$$\hat{a}^+\phi_n = \sqrt{n+1}\,\phi_{n+1}$$

[Op][function] = [Const][function]

[Op][function] = [Const][function]

$$\hat{H} = \hbar\omega(\hat{n} + \tfrac{1}{2})$$

$$\langle \phi_{n'} | x | \phi_n \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \langle \phi_{n'} | \hat{a} + \hat{a}^+ | \phi_n \rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega_0}} \left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

$$\frac{d}{dt}\langle \hat{a} \rangle \stackrel{?}{=} \cancel{\left\langle \frac{\partial}{\partial t}\hat{a} \right\rangle} + \frac{1}{i\hbar}\langle [\hat{a}, \hat{H}] \rangle = \frac{1}{i\hbar}\langle [\hat{a}, \hbar\omega_0 (\hat{a}^+\hat{a} + \tfrac{1}{2})] \rangle$$

$$\frac{d}{dt}\int \psi^*(x,t)\, \hat{a}\, \psi(x,t)\, dx$$

$$= \frac{\hbar\omega_0}{i\hbar} \langle [\hat{a}, \hat{a}^+\hat{a}] \rangle = -i\omega_0 \langle [\hat{a}, \hat{a}^+\hat{a}] \rangle$$

$[\hat{a}, \hat{a}^+ \hat{a}]\phi_n$ $\quad [\hat{a}, \hat{a}^+\hat{a}] = \hat{a}$ $\quad [a, \hat{a}^+]\phi_n = \hat{a}\hat{a}^+\phi_n - \hat{a}^+\hat{a}\phi_n$

$(\hat{a}, \hat{a}^+\hat{a})\phi_n - \hat{a}^+\hat{a}\hat{a}\phi_n$

$= (n+1)\phi_n - n\phi_n$

$= a^n n\phi_n - (n-1)\hat{a}\phi_n$

$= \phi_n$

$= \hat{a}\phi_n$

$[\hat{a}, \hat{a}^+] = 1$

4. $\boxed{\frac{d}{dt}\langle \hat{a}\rangle = -i\omega_0\langle \hat{a}\rangle}$

$\langle \hat{a}\rangle = A e^{-i\omega_0 t}$ $\Longleftarrow$ not hermitian
does not correspond to
physical observable

$[\hat{a}, \hat{a}^+] = 1$

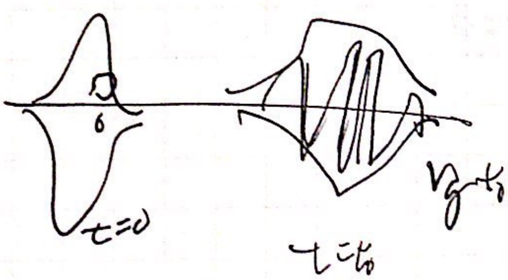
$\boxed{\frac{d}{dt}\langle \hat{a}^+\rangle = +i\omega_0\langle \hat{a}^+\rangle}$

$\langle \hat{a}^+\rangle = B e^{i\omega_0 t}$

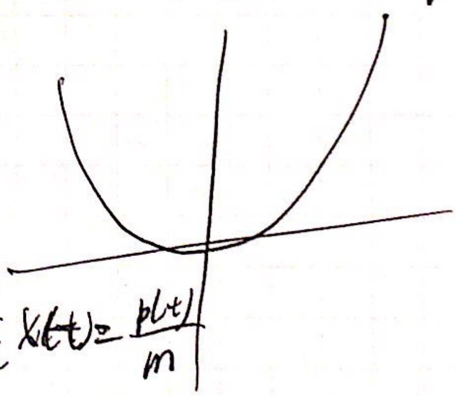
$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2$ $\quad i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 x^2\right)\psi(x,t)$

$= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{1}{2}m\omega$

Free space $\psi(x,t)$



. SHO $\quad V(x) = \frac{1}{2}m\omega_0^2 x^2$



$\frac{d}{dt}\langle x\rangle = \frac{\langle \hat{p}\rangle}{m}$ $\quad \Longleftrightarrow \quad \frac{d}{dt}X(t) = \frac{p(t)}{m}$

$\frac{d}{dt}\langle \hat{p}\rangle = -m\omega_0^2\langle x\rangle$

$\frac{d}{dt}p(t) = -m\omega_0^2 X(t)$

$$\frac{d}{dt}\langle (x-\langle x\rangle)^2\rangle = \frac{\langle (x-\langle x\rangle)(\hat{p}-\langle\hat{p}\rangle) + (\hat{p}-\langle p\rangle)(x-\langle x\rangle)\rangle}{m}$$

$$\frac{d}{dt}\langle(\hat{p}-\langle\hat{p}\rangle)^2\rangle = -m\omega_0^2\langle (x-\langle x\rangle)(\hat{p}-\langle\hat{p}\rangle) + (\hat{p}-\langle\hat{p}\rangle)(x-\langle x\rangle)\rangle$$

$$\frac{(\Delta p)^2}{2m} = \frac{1}{2}m\omega_0^2(\Delta x)^2$$

$$\frac{\langle\hat{p}^2\rangle}{2m} + \frac{1}{2}m\omega_0^2\langle x^2\rangle$$

$$(\Delta v)^2 = \frac{(\Delta p)^2}{m^2}$$

Free Space

if $\frac{d}{dt}(\Delta x)=0$

$$\frac{d}{dt}\langle x\rangle = \frac{\langle\hat{p}\rangle}{m} \qquad \frac{d}{dt}\langle\hat{p}\rangle = 0 \qquad (\Delta x)^2 = (\Delta x)^2\Big|_{t=0} + \frac{2}{m^2}(\Delta p)^2\frac{t^2}{2}$$

$$\frac{d^2}{dt^2}(\Delta x)^2 = \frac{2}{m^2}(\Delta p)^2 \qquad (\Delta p) = (\Delta p)\Big|_{t=0}$$

$$\frac{d}{dt}(\Delta p) = 0 \qquad (\Delta p) = (\Delta p)\Big|_{t=0}$$

$$\Psi(x,t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-i\theta(t)}\, e^{\frac{ip(t)(x-X(t))}{\hbar}}\, e^{-\frac{1}{2}\frac{m\omega_0}{\hbar}(x-X(t))^2}$$

$$\langle \hat{H}\rangle = \left\langle \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 x^2\right\rangle = \frac{1}{2}\hbar\omega_0 + \frac{p^2(t)}{2m} + \frac{1}{2}m\omega_0^2 X^2(t)$$

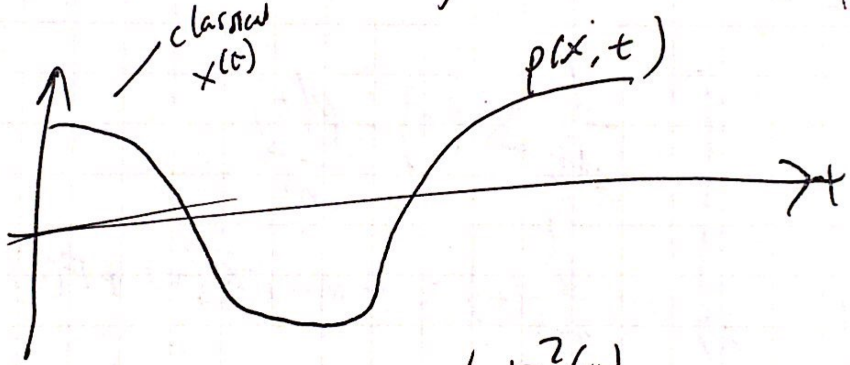
$$\underbrace{\frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega_0^2(\Delta x)^2}_{\frac{1}{2}\hbar\omega_0} + \underbrace{\frac{p^2(t)}{2m} + \frac{1}{2}m\omega_0^2 X^2(t)}_{\text{classical energy}}$$

= zero point energy

$$\hat{H} = \hbar\omega_0\left(\hat{n} + \frac{1}{2}\right)$$

$$\langle \hat{n}\rangle = \frac{\frac{p^2(t)}{2m} + \frac{1}{2}m\omega_0^2 X^2(t)}{\hbar\omega_0}$$

$$\langle \hat{H}\rangle = \hbar\omega_0\left(\langle \hat{n}\rangle + \frac{1}{2}\right)$$



$t=0$ $X(t)=X$

$p(t)=0$

$$\left(\frac{p^2(t)}{2m} + \frac{1}{2}m\omega_0^2 X^2(t)\right)\Bigg|_{t=0}$$

$$= \frac{1}{2}m\omega_0^2 X_{max}^2 = \hbar\omega_0\langle \hat{n}\rangle$$

$$X_{max}^2 = \frac{2\hbar}{m\omega_0}\langle \hat{n}\rangle$$

$$X_{max} = \sqrt{\frac{2\hbar}{m\omega_0}\langle \hat{n}\rangle}$$

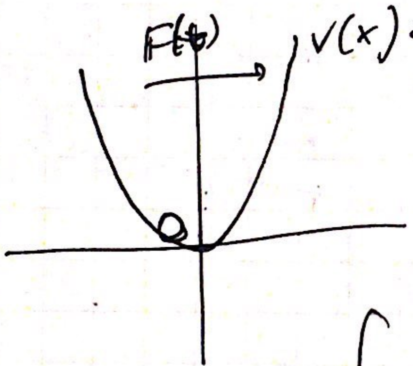
$$\frac{1}{2}m\omega_0^2(\Delta x)^2 = \frac{1}{4}\hbar\omega_0$$

$$(\Delta x)^2 = \frac{1}{2}\frac{\hbar}{m\omega_0} \qquad \Delta x = \sqrt{\frac{1}{2}\frac{\hbar}{m\omega_0}}$$

$$\frac{\text{Signal}}{\text{noise}} = \frac{\sqrt{\frac{2\hbar}{m\omega_0}\langle\hat{n}\rangle}}{\sqrt{\frac{\hbar}{2m\omega_0}}} = 2\sqrt{\langle\hat{n}\rangle}$$

this form is solvable

$$\underbrace{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2x^2 - F(t)\,x\right)\psi(x,t)}_{\text{quantum mechanical problem? Shows up everywhere}}$$



$V(x) = \frac{1}{2}\omega_0^2x^2$

This is ridiculously important!

$$\frac{d}{dt}X(t) = \frac{P(t)}{m}$$

$$\frac{d}{dt}P(t) = -m\omega_0^2X(t) + F(t)$$

$$\hbar\frac{d}{dt}\theta(t) = \frac{1}{2}\hbar\omega_0 - \frac{P^2(t)}{2m} + \frac{1}{2}m\omega_0^2X^2 - F(t)X(t)$$

figure what this is

also at quiz will be an exam?