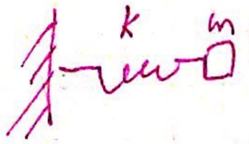
Lecture 11: 9/30

From last time

Classical SHO



$k = m\omega_0^2$

$V = \frac{1}{2} m \omega_0^2 x^2$

$$\frac{d}{dt} x(t) = \frac{p(t)}{m}$$

$$\frac{d}{dt} p(t) = -m\omega_0^2 x(t)$$

$$E = \frac{p^2(t)}{2m} + \frac{1}{2} m \omega_0^2 x^2(t)$$

Quantum SHO

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 x^2$$
$$= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega_0^2 x^2$$

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \left\{ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega_0^2 x^2 \right\} \psi(x,t)$$

$$\psi(x,t) = \psi(x) e^{-iEt/\hbar} \Rightarrow E\psi(x) = \left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega_0^2 x^2 \right\} \psi(x)$$

$$\frac{E}{\frac{1}{2}\hbar\omega_0} \psi = \frac{-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x)}{\frac{1}{2}\hbar\omega_0} + \frac{\frac{1}{2} m \omega_0^2 x^2 \psi(x)}{\frac{1}{2}\hbar\omega_0}$$

$$\psi(y) = e^{-y^2/2} [1 + a_2 y^2 + a_4 y^4 \ldots]$$

$$\Rightarrow \epsilon_n = 2n+1$$

$$\Rightarrow \epsilon \psi(y) = \frac{-d^2}{dy^2} \psi(y) + y^2 \psi(y)$$

$$y = \sqrt{\frac{m\omega_0}{\hbar}}\, x \qquad \epsilon = \frac{E}{\frac{1}{2}\hbar\omega_0}$$

$$\phi_n(y) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^n n!}} e^{-y^2/2} H_n(y)$$

$\phi_1 \ldots$ etc.

$$\langle \phi_{n'} | y | \phi_n \rangle = \int_{-\infty}^{\infty} \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^{n'} n'!}} e^{-y^2/2} H_{n'}(y)\, y\, \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^n n!}} e^{-y^2/2} H_n(y)\, dy$$

$$\hat{H} = -\frac{d^2}{dy^2} + y^2 = y^2 - \frac{d^2}{dy^2}$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\left(y + \frac{d}{dy}\right)\left(y - \frac{d}{dy}\right) f(y) = \left(y + \frac{d}{dy}\right)\left(yf - \frac{d}{dy} f\right) = y^2 f - y\frac{d}{dy}f + \frac{d}{dy}(yf) - \frac{d^2}{dy^2}f$$

$$= \left(-y^2 - \frac{d^2}{dy^2}\right) f(y) + \cancel{\left(\frac{d}{dy}y - y\frac{d}{dy}\right)} f$$

$$\left(y - \frac{d}{dy}\right)\left(y + \frac{d}{dy}\right) = \left(y^2 - \frac{d^2}{dy^2}\right) - 1$$

$$\left(y + \frac{d}{dy}\right)\left(y - \frac{d}{dy}\right) = \left(y^2 - \frac{d^2}{dy^2}\right) + 1$$

$$\left(y^2 - \frac{d^2}{dy}\right) = \frac{1}{2}\left(y - \frac{d}{dy}\right)\left(y + \frac{d}{dy}\right) + \frac{1}{2}\left(y + \frac{d}{dy}\right)\left(y - \frac{d}{dy}\right)$$

$$\left(y - \frac{d}{dy}\right) \underbrace{e^{-y^2/2}}_{\sim \phi_0(y)} = y e^{-y^2/2} + y e^{-y^2/2} = \underbrace{2y e^{-y^2/2}}_{\sim \phi_1(y)}$$

$$\left(y - \frac{d}{dy}\right) \underbrace{y e^{-y^2/2}}_{\sim \phi_1(y)} = y^2 e^{-y^2/2} - e^{-y^2/2} + y^2 e^{-y^2/2} = \underbrace{(2y^2 - 1) e^{-y^2/2}}_{\sim \phi_2(y)}$$

$$\left(y - \frac{d}{dy}\right) \underbrace{(2y^2 - 1) e^{-y^2/2}}_{\sim \phi_2(y)} = \ldots \sim \phi_3(y)$$

$$\left(y - \frac{d}{dy}\right)\phi_0(y) = \sqrt{2}\sqrt{1}\,\phi_1(y)$$

$$\left(y - \frac{d}{dy}\right)\phi_1(y) = \sqrt{2}\sqrt{2}\,\phi_2(y)$$

$$\left(y - \frac{d}{dy}\right)\phi_2(y) = \sqrt{2}\sqrt{3}\,\phi_3(y)$$

$$\left(y - \frac{d}{dy}\right)\phi_n(y) = \sqrt{2}\sqrt{n+1}\,\phi_{n+1}(y)$$

[op][function] = [const][function]

$\hat{H}\Psi = E\Psi$

$\hat{p}\, e^{ikx} = \hbar k\, e^{ikx}$

[op][function] = [const][function]

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) \qquad \left(y+\frac{d}{dy}\right)e^{-y^2/2} = ye^{-y^2/2} - ye^{-y^2/2} = 0$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) \qquad \left(y+\frac{d}{dy}\right)\underbrace{ye^{-y^2/2}}_{\hat{\phi}_1(y)} = y^2e^{-y^2/2} - y^2e^{-y^2/2} + e^{-y^2/2} = e^{-y^2/2} \sim \phi_0(y)$$

$$\left(y+\frac{d}{dy}\right)\phi_n(y) = \sqrt{2}\sqrt{n}\,\phi_{n-1}(y)$$

$$\hat{a}^+\phi_n(y) = \sqrt{n+1}\,\phi_{n+1}(y)$$

$$\hat{a}\phi_n(y) = \sqrt{n}\,\phi_{n-1}(y)$$

☆ To do: Really understand what a Hermitian is

$$\int_{-\infty}^{\infty}(\hat{H}\psi)^*\psi\,dx = \int_{-\infty}^{\infty}\psi^*(H\psi)\,dx$$

$$\int_{-\infty}^{\infty}(\hat{a}\phi_n')^*\phi_n\,dy = \int_{-\infty}^{\infty}\phi_{n'}^*(\hat{a}^+\phi_n)\,dx$$

$$\left(y^2+\frac{d^2}{dy^2}\right)\phi_n(y) = (\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})\phi_n(y) = \hat{a}\sqrt{n+1}\,\phi_{n+1} + \hat{a}^+\sqrt{n}\,\phi_{n-1}$$

$$= (n+1)\phi_n + n\phi_n = (2n+1)\phi_n = \epsilon_n\phi_n$$

$$\langle \phi_{n'}|y|\phi_n\rangle = \int_{-\infty}^{\infty}\phi_{n'}^*(y)\,y\,\phi_n(x)\,dy = \int \phi_{n'}^*\frac{(\hat{a}+\hat{a}^+)}{\sqrt{2}}\phi_n\,dy$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(y+\frac{d}{dy}\right)$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(y-\frac{d}{dy}\right)$$

$$(\hat{a}+\hat{a}^+) = \sqrt{2}\,y$$

$$= \frac{1}{\sqrt{2}}\int\phi_{n'}^*\sqrt{n}\,\phi_{n-1}\,dy + \frac{1}{\sqrt{2}}\int\phi_{n'}^*\sqrt{n+1}\,\phi_{n+1}\,dy$$

$$= \frac{\sqrt{n}}{\sqrt{2}}$$

$$\therefore \frac{\sqrt{n}}{\sqrt{2}}\delta_{n',n-1} + \frac{\sqrt{n+1}}{\sqrt{2}}\delta_{n',n+1}$$

$$\int_{-\infty}^{\infty}\frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^{n'}n'!}}e^{-y^2/2}H_{n'}(y)\,y\,\frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^n n!}}e^{-y^2/2}H_n(y)\,dy$$

$$= \frac{\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}}{\sqrt{2}}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x + \sqrt{\frac{\hbar}{m\omega_0}}\frac{d}{dx}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x + i\sqrt{\frac{1}{\hbar m\omega_0}}\,\hat{p}\right)$$

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x - \sqrt{\frac{\hbar}{m\omega_0}}\frac{d}{dx}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x - i\sqrt{\frac{1}{\hbar m\omega_0}}\,\hat{p}\right)$$

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\hat{a} + \hat{a}^{\dagger}\right)$$

$$\hat{p} = \sqrt{\frac{\hbar m\omega_0}{2}}\left(\frac{\hat{a} - \hat{a}^{\dagger}}{i}\right) \qquad \langle \phi_n | x | \phi_n \rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\sqrt{n}\,\delta_{n,n-1} + \sqrt{n+1}\,\delta_{n,n+1}\right)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 = \frac{-1}{2m}\left(\frac{\hbar m\omega_0}{2}\right)\left(\hat{a}^2 - \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} + (\hat{a}^{\dagger})^2\right) + \frac{1}{2}m\omega_0^2\left(\frac{\hbar}{2m\omega_0}\right)\left(\hat{a}^2 + \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + (\hat{a}^{\dagger})^2\right)$$

$$\frac{\hbar\omega_0}{4}$$

$$\hat{a}^{\dagger}\hat{a}\,\phi_n = \hat{a}^{\dagger}\sqrt{n}\,\phi_{n-1} = n\,\phi_n$$

$$\hat{n} = \hat{a}^{\dagger}\hat{a} \qquad \hat{n}|n\rangle = n|n\rangle$$

$$\hat{a}\hat{a}^{\dagger} = \hat{n} + 1$$

Idea: Creation of Annihilation Operators for the quantum Simple Harmonic Oscillator