Lecture 10

9/27

$\psi_I(x) = ce^{-ikx}$

$\psi_{II}(x) = ae^{-\alpha x} + be^{\alpha x}$

$\psi_{III}(x) = \cos(kx)$

$\psi_{IV}(x) = ae^{\alpha x} + be^{-\alpha x}$

$\psi_V(x) = ce^{ikx}$

make it non-hermitian

Hermetian prob.

$$\int_{-\infty}^{\infty} (\hat{H}\psi)^* \psi \, dx = \int_{-\infty}^{\infty} \psi^* (\hat{H}\psi) \, dx$$

The problem

$$\int_{-\infty}^{\infty} (\hat{H}\psi)^* \psi \, dx = \int_{-\infty}^{\infty} \psi^* (\hat{H}\psi) \, dx + i\,4k|c|^2 \frac{\hbar^2}{2m}$$

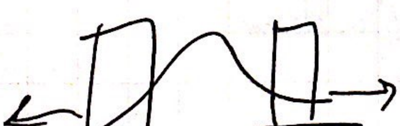
match bdy conditions

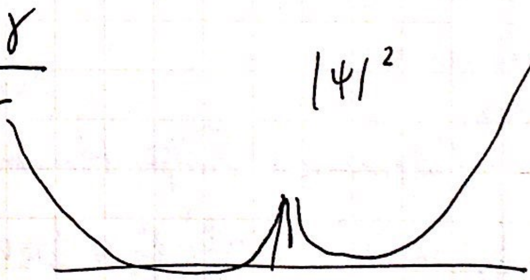
$$\frac{(\alpha - ik) - (\alpha + ik)e^{-2\alpha d}}{(\alpha - ik) + (\alpha + ik)e^{-2\alpha d}} = \frac{k}{\alpha}\tan\left(\frac{kL}{2}\right)$$

$$\psi(x,t) = e^{-iEt/\hbar} = e^{-i\frac{Re\{E\}t}{\hbar}} e^{-\frac{\gamma t}{2}} \qquad E = \frac{\hbar k^2}{2m} \qquad E = \frac{-\hbar^2 \alpha^2}{2m} + V_0$$

$$P(x,t) = |\psi(x,t)|^2 = |\psi(x)|^2 e^{-\gamma t}$$

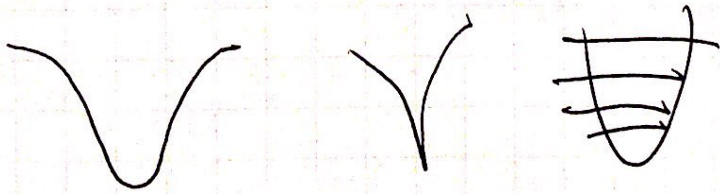
$$E = Re\{E\} - \frac{i\hbar\gamma}{2}$$





Time independent Schrödinger eq'n

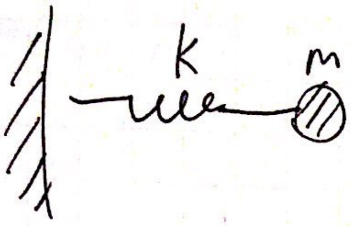
$$E\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) \qquad \psi(x,t) = \sum_i a_i \phi_i(x) e^{-iE_i t/\hbar}$$



$E_n = n\hbar\omega_0 \qquad E_n = (n+\frac{1}{2})\hbar\omega_0$

Simple Harmonic Oscillator

Classical version



$$\frac{d}{dt}x(t) = \frac{P(t)}{m} \qquad \frac{d^2}{dt^2}x(t) = \frac{1}{m}\frac{dp(t)}{dt}$$

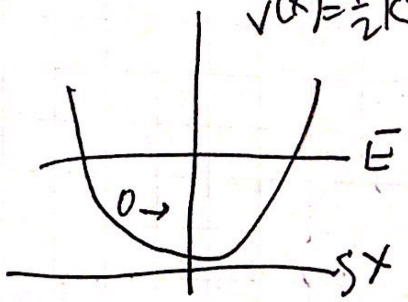
$$= -\frac{k}{m}x(t)$$

$$\frac{d}{dt}p(t) = -kx(t)$$

$$\frac{d^2}{dt^2}x(\cancel{t}) = -\omega_0^2 x(t) \Rightarrow \omega_0^2 = \frac{k}{m}$$

$$E = \frac{p^2(t)}{2m} + \frac{1}{2}kx^2(t) \qquad x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$V(x) = \frac{1}{2}kx^2$



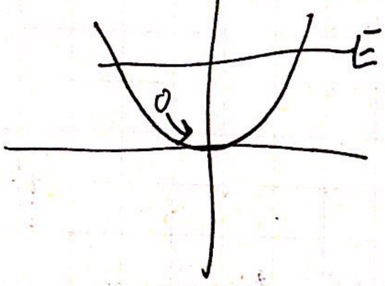
$$k = m\omega_0^2$$

$$\frac{d}{dt}x(t) = \frac{p(t)}{m}$$

$$\frac{d}{dt}p(t) = -m\omega_0^2 x(t)$$

$$E = \frac{p^2(t)}{2m} + \frac{1}{2}m\omega_0^2 x^2(t)$$

$V(x) = \frac{1}{2}m\omega_0^2 x^2$



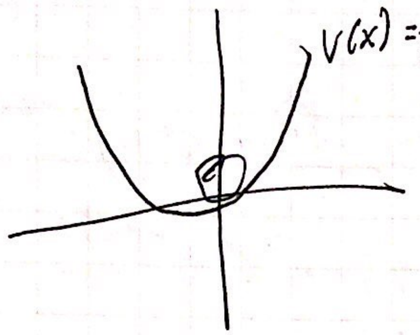
Quantum Mechanical Simple Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + \frac{1}{2}m\omega_0^2 x^2\,\Psi(x,t)$$

$$\Psi(x,t) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega_0^2 x^2\psi(x)$$

Ground State



Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Back of the envelope argument

$$\int_{-\infty}^{\infty} \psi^* \left(E\psi = \hat{H}\psi = \frac{\hat{p}^2}{2m}\psi + \frac{1}{2} m \omega_0^2 x^2 \psi \right) dx$$

$$\longrightarrow E \underbrace{\langle \psi | \psi \rangle}_{=1} = \frac{\langle \hat{p}^2 \rangle}{2m} + \frac{1}{2} m \omega_0^2 \langle x^2 \rangle = E$$

$$\langle x^2 \rangle = \langle x \rangle^2 + \langle (x - \langle x \rangle)^2 \rangle = \underbrace{\langle x \rangle^2}_{\to 0} + (\Delta x)^2$$

$$\langle \hat{p}^2 \rangle = \langle \hat{p} \rangle^2 + \langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle = \underbrace{\langle \hat{p} \rangle^2}_{\to 0} + (\Delta p)^2$$

Ehrenfest theorem gives

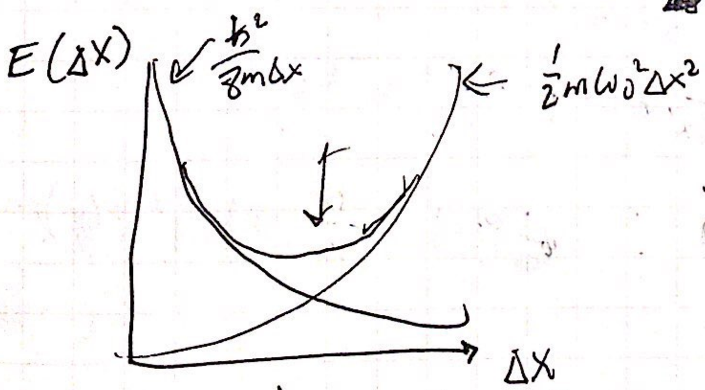
$$\frac{d}{dt}\langle x \rangle = \frac{\langle \hat{p} \rangle}{m} \qquad \frac{d}{dt}\langle \hat{p} \rangle = -m\omega_0^2 \langle x \rangle$$

$$E = \frac{\langle \hat{p}^2 \rangle}{2m} + \frac{1}{2} m \omega_0^2 \langle x^2 \rangle = \frac{\Delta p^2}{2m} + \frac{1}{2} m \omega_0^2 \Delta x^2$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E[\Delta x] = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2} m \omega_0^2 \Delta x^2 \qquad \Delta p = \frac{\hbar}{2\Delta x} \text{ approx.}$$

Recall
Gaussian $\Delta x \Delta p = \frac{\hbar}{2}$
w/ hyperbolic secant, $\Delta x \Delta p = \hbar \pi / 6 \approx \hbar/2$



$$\frac{d}{d\Delta x} E[\Delta x] = 0$$

$$\frac{d}{d\Delta x} E[\Delta x] = \frac{-\hbar^2}{4\Delta x^3} + m\omega_0^2 \Delta x = 0$$

$$\Delta x^4 = \frac{\hbar^2}{4m^2\omega_0^2} \Rightarrow \Delta x_{opt} = \sqrt{\frac{\hbar}{2m\omega_0}}$$

$$E[\Delta x_{opt}] = \frac{\hbar^2}{8m\Delta x_{opt}^2} + \frac{1}{2} m \omega_0^2 \Delta x_{opt}^2$$

$$= \frac{1}{4}\hbar\omega_0 + \frac{1}{4}\hbar\omega_0 = \frac{1}{2}\hbar\omega_0 = \text{Planck's 1911 zero pt estimate}$$

$$E\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega_0^2 x^2\psi(x)$$

$$\psi(x) = e^{-\beta x^2/2}$$

$$\frac{d}{dx}\psi = -\beta x e^{-\beta x^2/2}$$

$$\frac{d^2}{dx^2}\psi = \left((\beta x)^2 - \beta\right)e^{-\beta x^2/2}$$

$$E e^{-\beta x^2/2} = \frac{-\hbar^2}{2m}\left((\beta x)^2 - \beta\right)e^{-\beta x^2/2} + \frac{1}{2}m\omega_0^2 x^2 e^{-\beta x^2/2}$$

$$E = \frac{\hbar\beta}{2m} \qquad \frac{\hbar^2\beta^2}{2m} = \frac{1}{2}m\omega_0^2$$

$$\Rightarrow \beta^2 = \frac{m^2\omega_0^2}{\hbar^2} \Rightarrow \beta = \frac{m\omega_0}{\hbar}$$

agrees w/ ground state solution

$$\psi(x) = Ce^{-\frac{m\omega_0}{2\hbar}x^2}$$

$$E = \frac{1}{2}\hbar\omega_0$$

$$V(x) = \frac{1}{2}m\omega_0^2 x^2$$

$$= Ce^{-\frac{\frac{1}{2}m\omega_0^2 x^2}{\hbar\omega_0}}$$

← interesting because we're going to do SHM equations not in space

$$\int_{-\infty}^{\infty}|C|^2 e^{\frac{-m\omega_0 x^2}{\hbar}}dx = \frac{|C|^2\sqrt{\pi}}{\sqrt{\frac{m\omega_0}{\hbar}}} = 1$$

$$\Rightarrow C = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}$$

mathematical physicists solve dfq for diff. model.
schrödinger first person to do it for SHO

$$\frac{E\psi(x)}{\frac{1}{2}\hbar\omega_0} = \frac{\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)}{\frac{1}{2}\hbar\omega_0} + \frac{\frac{1}{2}m\omega_0^2 x^2\psi(x)}{\frac{1}{2}\hbar\omega_0}$$

$$\epsilon = \frac{E}{\frac{1}{2}\hbar\omega_0} \qquad \epsilon\psi(x) = \frac{-\hbar}{m\omega_0}\frac{d^2}{dx^2}\psi(x) + \frac{m\omega_0}{\hbar}x^2\psi(x)$$

$$y^2 = \frac{m\omega_0}{\hbar}x^2 \qquad y = \sqrt{\frac{m\omega_0}{\hbar}}x \qquad \epsilon\psi(y) = \frac{-d^2}{dy^2}\psi(x) + y^2\psi(y)$$

Even solutions

$$\psi(y) = e^{-y^2/2}\left[1 + a_2y^2 + a_4y^4 + \dots\right]$$

$$\frac{d}{dy}\psi = -y\, e^{-y^2/2}[\quad] + e^{-y^2/2}\left[2a_2y + 4a_4y^3 + \dots\right]$$

$$\frac{d^2}{dy^2}\psi = y^2 e^{-y^2/2}[1 + \dots]$$
$$- e^{-y^2/2}[1 + \dots]$$
$$- y e^{-y^2/2}[2a_2 y + 4a_4 y^3 + \dots]$$
$$+ e^{-y^2/2}[2a_2 + 4\cdot 3 a_4 y^2 + 6.5 a_6 y^4 + \dots]$$

$$(\epsilon - 1) e^{-y^2/2}[1 + \quad] = +2y \cancel{e^{-y^2/2}}[2a_2 y + 4a_4 y^3 \dots]$$
$$- e^{-y^2/2}[2\cdot 1 a_2 + 4\cdot 3 a_4 y^2 + \dots]$$

$$(\epsilon - 1) = -2a_2$$
$$(\epsilon - 1) = -4\cdot 3 a_4 + 4a_2$$
$$(\epsilon - 1)a_4 = -(n+2)(n+1)a_{n+2} + 2n a_n$$
$$(\epsilon - 2n - 1)a_n = -(n+2)(n+1)a_{n+2}$$
$$a_{n+2} = -\frac{(\epsilon - 2n - 1)a_n}{(n+2)(n+1)}$$

2 term recursion relation

$$\boxed{\epsilon = 2n+1}$$

$$E_n = \frac{1}{2}\hbar \omega_0 \epsilon_n = (n + \tfrac{1}{2})\hbar\omega_0$$ Planck's energy expression

$$\phi_0(y) = \frac{1}{\pi^{1/4}} e^{-y^2/2} \qquad \epsilon = 1 \qquad \psi(y) = e^{-y^2/2}(a_1 y + a_3 y^3 + \dots)$$

$$\phi_2(y) = \frac{1}{2^{1/2}} e^{-y^2/2}(1 - 2y^2) \qquad \epsilon = 5 \qquad \epsilon_0 = 1 \qquad \phi_1(y) = \frac{2^{1/2}}{\pi^{1/4}} y e^{-y^2/2}$$

$$\phi_n(y) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^n n!}} e^{-y^2/2} H_n(y) \qquad \epsilon_1 = 3$$

Hermitian polynomial

$$\epsilon_n = 2n+1$$

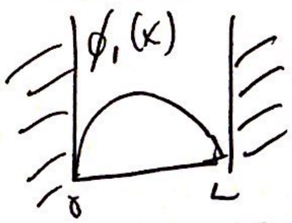
QM ~~Simple~~ Harmonic Oscillator

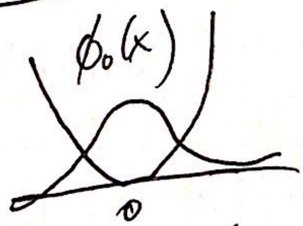
$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega_0^2x^2\psi(x)$$

$$\phi_n(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\, e^{-\frac{1}{2}\frac{m\omega_0 x^2}{\hbar}}\, H_n\!\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x\right)$$

$$E_n = \hbar\omega_0\left(n+\frac{1}{2}\right)$$

Square Well - ~~S~~HO



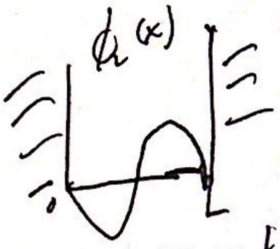


quadratic

$E_1 = \frac{\hbar^2\pi^2}{2mL^2}$

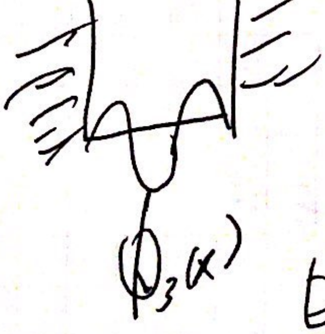
$E_0 = \frac{\hbar\omega_0}{2}$

linear



$E_2 = \frac{\hbar^2\pi^2}{2mL^2}(4)$

$E_1 = \frac{3}{2}\hbar\omega_0$



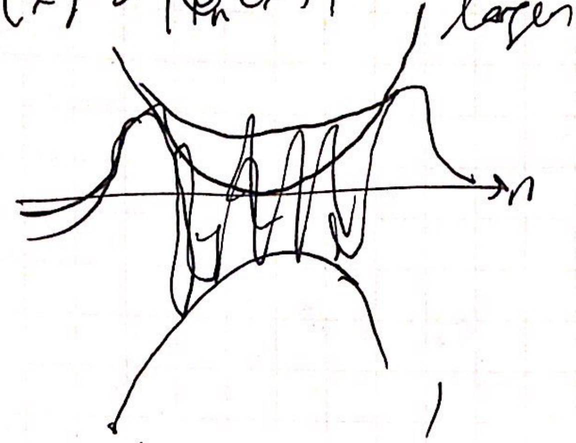
$\phi_2(x)$

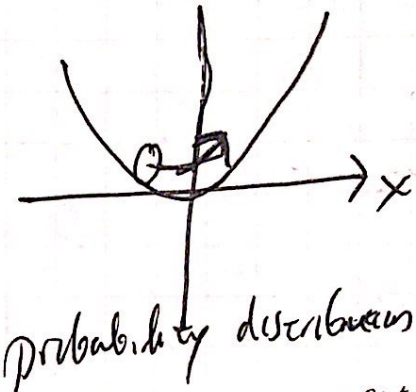
$E_2 = \frac{5}{2}\hbar\omega_0$

$\phi_3(x)$

$E_3 = \frac{\hbar^2\pi^2}{2mL^2}9$

$P_n(x) = |\phi_n(x)|^2$ larger





$$\phi_n(x) \sim \frac{1}{(E - V(x))^{1/4}}$$

probability distribution?
what measurement?
time in each spot.

$$\int_{-\infty}^{\infty} \phi_n(x)\, \phi_n(x)\, dx$$

what it add x^2

$$P(x) \sim \frac{1}{v(x)} \sim \frac{1}{\sqrt{E - V(x)}}$$

$$E = \frac{p^2}{2m} + V(x) = \frac{1}{2} m v^2 + V(x)$$

$$\frac{1}{2} m v^2 = E - V(x)$$

$$v \sim \sqrt{E - V(x)}$$

$$= \int_{-\infty}^{\infty} \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\frac{1}{2}\frac{m\omega_0 x^2}{\hbar}} H.$$

$$\left(\sqrt{\frac{m\omega_0}{\hbar}}\, x\right) \left(\ \right)^{1/4} \frac{1}{\sqrt{\ }} e^{-\frac{1}{2}(\,)x^2}$$

$$H_n(\,)\, dx = \begin{cases} \frac{1}{\hbar} & n = n' \\ 0 & n \neq n' \end{cases}$$

$$E_n = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m \omega_0^2 \langle x^2 \rangle$$

what it add x^2?

headache

$$= \left(n + \frac{1}{2}\right)\hbar\omega_0$$

$$\langle x^2 \rangle = \frac{(n + \frac{1}{2})\hbar\omega_0}{\frac{1}{2} m \omega_0^2} = \left(n + \frac{1}{2}\right)\frac{\hbar}{m\omega_0}$$

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