Lecture 15: Phase Plane Portraits

3/11/19

- saddles, nodes
- sources, sinks
- centers, spiral

} trajectory

Last Time:

$\dot{\vec{x}} = A\vec{x}$ solns $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$ 2x2 case

numbers that we get: eigenvalues → solve characteristic polynomial

$\det(A-\lambda I)=0$ eigenvalues; $(A-\lambda_i I)\vec{x_i}=\vec{0}$

Romeo + Juliet

$x(t)$ = amount Juliet loves Romeo   what happens as $t \to \infty$

$y(t)$ = amount Romeo loves Juliet

Ex. 1 $\dot{x} = y$ Juliet is responsive when she realizes that Romeo is interested

$\dot{y} = 100x$ Romeo is hypersensitive

$\dot{\vec{x}} = (A\vec{x})$ $A = \begin{pmatrix} 0 & 1 \\ 100 & 0 \end{pmatrix}$ $|I\lambda - A| = \begin{vmatrix} -\lambda & 1 \\ 100 & -\lambda \end{vmatrix} = \lambda^2 - 100 = 0$

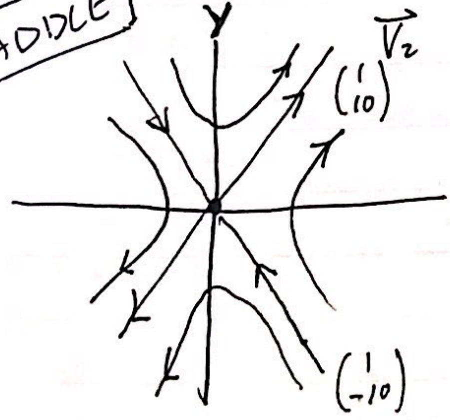
$\lambda = \pm 10$

$\lambda_1 = -10$ $\begin{pmatrix} 10 & 1 \\ 100 & 10 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ -10 \end{pmatrix}}_{\vec{V}_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\boxed{\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ -10 \end{pmatrix} e^{-10t} + C_2 \begin{pmatrix} 1 \\ 10 \end{pmatrix} e^{10t}}$$

$\lambda_2 = 10$ $\begin{pmatrix} -10 & 1 \\ 100 & -10 \end{pmatrix} \begin{pmatrix} 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

SADDLE



no t on this map! instead t is an arrow on the trajectory.

As $t \to \infty$

$y > -10x_0$ $\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$y_0 + 10_0 > 0$

$\begin{pmatrix} x \\ y \end{pmatrix}$ resembles $C_2 \begin{pmatrix} 1 \\ 10 \end{pmatrix} e^{10t}$ $\boxed{C_2 > 0}$

because $e^{10t} >> e^{-10t}$

Ex 2

$A=\begin{pmatrix}-3 & 1\\ 1 & -3\end{pmatrix}$ $\dot{x}=-3x+y$ $\dot{y}=x-3y$

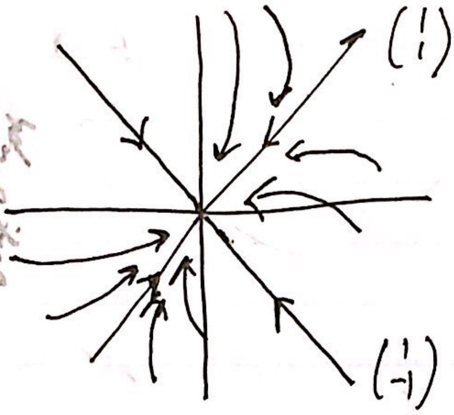
Juliet is responsive but has a fear of commitment

Romeo is similar

Recall from last lecture

$\vec{x}(t)=C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}+C_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t}$ $\lambda_1,\lambda_2=-2,-4$

not sure if this graph is right.



use eigenvectors as scaffolds

Which vector is dominant? as $t\to\infty$

$e^{-2t}\gg e^{-4t}$

$x(t)\to 0$

$y(t)\to 0$ no spark of interest survives.

$\boxed{\text{Node, nodal sink}}$

when $t\to-\infty$

$e^{-4t}\gg e^{-2t}$

Ex 3

$A=\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}$

$\begin{cases}\dot{x}=3x-y\\ \dot{y}=-x+3y\end{cases}$ Juliet is contrary (self reinforcing)

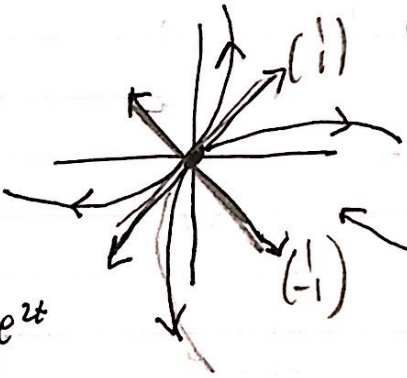
Romeo is similar

Time reversible $(t\to-t)$ of Ex. 2

$\vec{x}(t)=C_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+C_2\begin{pmatrix}1\\-1\end{pmatrix}e^{4t}$ ← largest as $t\to\infty$

$e^{4t}\gg e^{2t}$

$\boxed{\text{Nodal Source}}$



$t\to-\infty$

In ex.3, as $t\to\infty$ $\frac{x(t)}{y(t)}\to-1$

Ex 4

$A=\begin{pmatrix}0 & 100\\ -1 & 0\end{pmatrix}$ $\dot{x}=100y$ Juliet is hypersensitive

$\dot{y}=-x$ Romeo is contrary

$\begin{vmatrix}-\lambda & 100\\ -1 & -\lambda\end{vmatrix}=\lambda^2+100=0$

$\lambda=\pm10i$

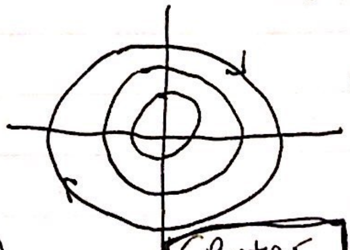
$\lambda=-10i$ $\begin{pmatrix}+10i & 100\\ -1 & 10i\end{pmatrix}\begin{pmatrix}10i\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

Draw one solution: $\begin{pmatrix}10\sin(10t)\\ \cos(10t)\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$

Solution: $C_1\begin{pmatrix}10i\\1\end{pmatrix}e^{-10it}\left(+C_2\begin{pmatrix}-10i\\1\end{pmatrix}e^{10it}\right)$

won't need for discussion

$\begin{pmatrix}10i\\1\end{pmatrix}[\cos(10t)-i\sin(10t)]=\begin{pmatrix}10\sin(10t)\\ \cos(10t)\end{pmatrix}+i\begin{pmatrix}10\cos(10t)\\ -\sin(10t)\end{pmatrix}$



$x^2=100\sin^2 10t+100y^2$ $100\cos^2(10t)$

$\boxed{\begin{matrix}x^2+10y^2=100\\ \text{ellipse}\end{matrix}}$

$\boxed{\text{Center}}$

same solution but with time shift

decide if clockwise or counterclockwise.

Warning: You can't decide on an orientation (counterclockwise vs. clockwise) Using the characteristic equation and eigenvalues.

Instead, look at the velocity vector.

i.e. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 100 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ 

clockwise

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$$\left(\begin{matrix} \ddot{x} + 4\dot{x} + 4x = 0 \\ -t, x(t) \end{matrix} \right)$$

↳ system, $x_1(t)$, $x_2(t)$

linear systems with constant coefficient

$$\begin{cases} \dot{x}_1 = 8x_1 + x_2 \\ \dot{x}_2 = 2x_1 + 7x_2 \end{cases} \qquad \underbrace{\begin{pmatrix} 8 & 1 \\ 2 & 7 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

↘ Matrix form

Find eigenvalue & eigenvectors of A.

$\det(|A - \lambda I|)$

→ $V \neq 0 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

V is an eigenvector with eigenvalue λ if

$Av = \lambda v$

$$= \begin{vmatrix} 8-\lambda & 1 \\ 2 & 7-\lambda \end{vmatrix} = (8-\lambda)(7-\lambda) - 2 = 56 - 15\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 15\lambda + 54$$

$$= (\lambda - 6)(\lambda - 9) \Rightarrow \lambda = 6, 9$$

$$\lambda = 6 \quad \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

check:

$$\underset{A}{\begin{pmatrix} 8 & 1 \\ 2 & 7 \end{pmatrix}} \underset{V}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} \underset{=}{=} \begin{pmatrix} -6 \\ 12 \end{pmatrix} = \underset{\lambda}{6} \underset{V}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} \quad \checkmark$$

$$\lambda = 9 \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{6t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 e^{9t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When it doesn't work...

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V_2 = 0 \cdot V \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ any non zero vector satisfies the equation

alternative method:

$B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$tr(B)=a+d$

$det(B)=ad-bc$

$det(B-\lambda I)$

$=\lambda^2-tr(B)\lambda+det(B)$

When eigenvalues come out imaginary:

$$\begin{cases} \dot{x}_1=-2x_1+2x_2 \\ \dot{x}_2=-x_1 \end{cases}$$

$$\underbrace{\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

$det(A-\lambda i)=0 \leftarrow$ characteristic polynomial

$tr(A)=-2$

$det(A)=2$

$\lambda^2+2\lambda+2=0$

$$\Rightarrow \lambda=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$$

$\lambda=-1+i \quad A=\lambda I$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

only need to do one eigenvector. The other is complex conjugate:

$\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$

$$C_1e^{(-1+i)t}\begin{pmatrix} 2 \\ 1+i \end{pmatrix}+C_2e^{(-1-i)t}\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

C_1 and C_2 are complex numbers.

Real Solutions

$$e^{(-1+i)t}\begin{pmatrix} 2 \\ 1+i \end{pmatrix}=e^{-t}e^{it}\begin{pmatrix} 2 \\ 1+i \end{pmatrix}=e^{-t}(\cos t+i\sin t)\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

Re

$$e^{-t}\begin{pmatrix} 2\cos t \\ \cos t-\sin t \end{pmatrix}$$

Im

$$e^{-t}\begin{pmatrix} 2\sin t \\ \cos t+\sin t \end{pmatrix}$$

$$C_1e^{-t}\begin{pmatrix} 2\cos t \\ \cos t-\sin t \end{pmatrix}+C_2e^{-t}\begin{pmatrix} 2\sin t \\ \cos t+\sin t \end{pmatrix}$$

Lecture 16: Phase Portraits (Continued)

3/13/19

- nodes, spirals
- trace-determinant plane
- stability
- structural stability (borderlines)

Last time

$\vec{x} = A\vec{x}$ $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - (-1)^2$

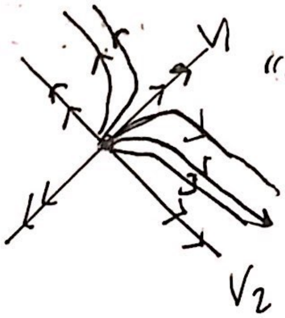
$= \lambda^2 - 6\lambda + 8$

Shortcut: trace(A) = 6   notice this is a minus ★★

det(A) = 8

$\lambda = +2, +4 \sim e^{2t}, e^{4t}$

$x(t) = c_1\vec{v}e^{2t} + c_2\vec{v}_2 e^{4t}$



"eigenlines"

$t \to +\infty$

real eigenvectors divide into 4 quadrants

$t \to \infty$ $e^{4t} \gg e^{2t}$

$t \to -\infty$ $e^{2t} \ll e^{4t}$

$\boxed{\text{Nodal Sarce}}$

Ex. 4 center

Ex. 5 $\dot{\vec{x}} = A\vec{x}$

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ contrary responsive

$\begin{vmatrix} -1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$

$\lambda = 1 \pm i$

$\begin{pmatrix} 1-(1-i) & i-1 \\ 1 & 1-(1-i) \end{pmatrix}$

$\lambda = 1 - i$ $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\underbrace{\begin{pmatrix} 1 \\ i \end{pmatrix}}_{\vec{v}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\cos(-t) = \cos t$

$\sin(-t) = -\sin(t)$

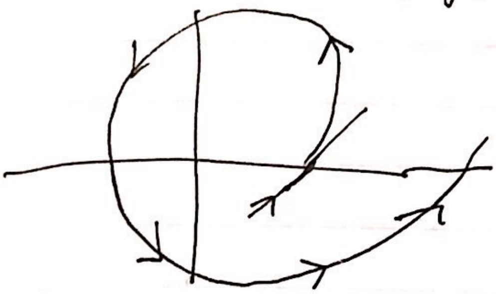
$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ i \end{pmatrix}e^{(1-i)t} + c_2\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{(1+i)t}$ $e^{it} \rightleftharpoons {}^{-it}$

$\begin{pmatrix} 1 \\ i \end{pmatrix}e^{(1-i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix}e^{t}(\underbrace{\cos t}_{\cos(-t)} - i\underbrace{\sin t}_{-\sin(t)}) = e^{t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + ie^{t}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

One solution is $e^{t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $e^{t}\cos t$, $e^{t}\sin t$

@ $\binom{1}{0}$ which way am I going? $\dot{\vec{x}} = A\binom{1}{0} = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\binom{1}{0} = \binom{1}{1}$

North east



Characteristic Polynomial:

$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$

Ex. 3:

$\lambda^2 - 6\lambda + 8 = 0$

$6 = \text{trace}\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}$ $8 = \det\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}$

SPIRAL : COMPLEX ROOTS $\lambda = s \pm i\omega$ $(\omega \neq 0)$

Node: λ_1, λ_2 same sign

Saddle: λ_1, λ_2 opposite sign (always unstable)

Nodes examples: $C_1\vec{V}_1e^{2t} + C_2\vec{V}_2e^{4t}$ unstable (source)

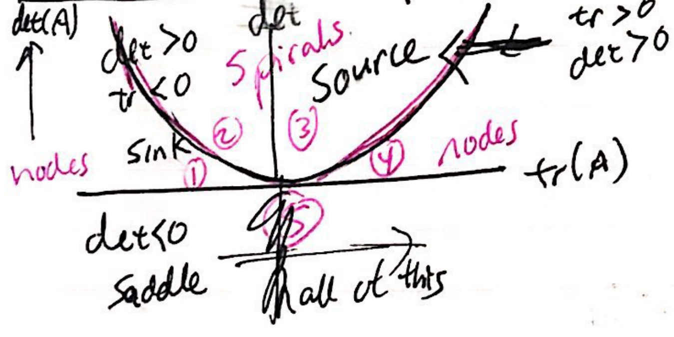
$C_1\vec{V}_1e^{-2t} + C_2\vec{V}_2e^{-4t}$ stable (sink)

Spirals: $e^{(1\pm i)t}$; $e^t \to \infty$ as $t \to \infty \Rightarrow$ unstable

e^{-t} is stable

#IMPORTANT BTW

trace-determinant plane



i.e. $\lambda^2 - 10\lambda + 80 = 0$

Spiral happens

$\overbrace{\text{complex roots} \Leftrightarrow (\text{trace}(A))^2 - 4\det A < 0}$

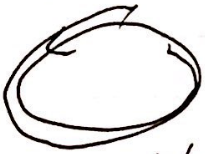
$\Updownarrow$

that ∪ $\boxed{\det A > \frac{1}{4}\text{trace}(A)^2}$

border: $\det(A) = \dfrac{\text{tr}(A)^2}{4}$

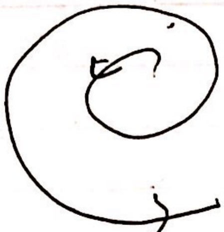
there are 5 regions

borderline between spiral sinks & spiral ~~[illegible]~~ sources is "center"



stable ~~[illegible]~~
Spiral: ~~[illegible]~~ Spiral
sink

center



unstable spiral =
~~S~~ Spiral source

Is the center stable?

NO

Structural Stability (not same a stability)

~~[illegible]~~ | DEF'N $x(t) \to 0$ all homogeneous solutions = stability |

If we change the coefficients (i.e. A) slightly
then the picture does not change ⇒ Fails exactly on borderlines

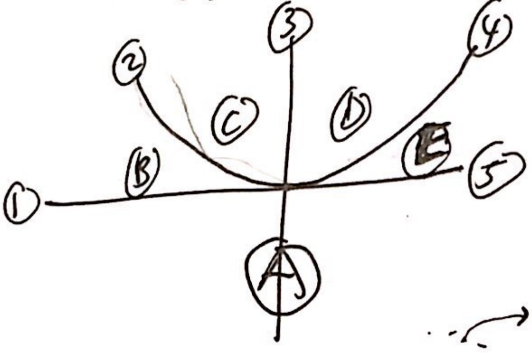
Lecture 17: Matrices, Geometric + Arithmetic

3/15/19

- borderlines in trace-determinant representation
- geometry of matrix multiplication
- solving $A\vec{x} = \vec{b}$ (elimination algorithm)

Review: $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$

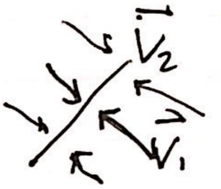


1, 2, 3, 4, 5 → (structurally unstable) borderlines

ABCDE → structurally stable regions.

Ⓐ Saddle: $\lambda_1 < 0$, $\lambda_2 > 0$

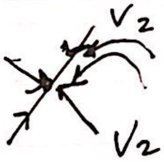
① $\lambda_1 < 0$ $\lambda_2 = 0$ "comb"



In these pictures, $\vec{V}_2$ dominates

$e^{\lambda_2 t} \gg e^{\lambda_1 t}$

Ⓑ Stable node



$\lambda_1 < 0$ $\lambda_2 < 0$

② $\lambda_1 = \lambda_2$ $e^{\lambda_1 t} = e^{\lambda_2 t}$ $\vec{v}_1$ and $\vec{v}_2$ collapse to one vector

Stable "degenerate node"

← drew very close to each other

another case $\vec{V}_1$ & $\vec{V}_2$ stay apart

Star node

Stable

② → Spiral Ⓒ

you visualize how to modify ②

Geometry of Matrix Multiplication

recall $\dot{x} = ix$ (Euler) → complex arithmetic $\frac{1}{a+bi}$ (polar form)

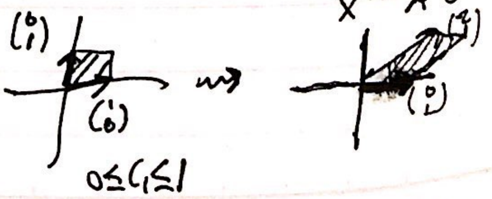
$\longleftrightarrow \dot{\vec{x}} = A\vec{x}$ $\vec{x}$ vectors A matrices → linear algebra $\left|\frac{1}{a+bi}\right|$

$A\vec{x} = \vec{b}$ & what do we mean by $\vec{x} = \frac{1}{A}\vec{b}$?

Matrix multiplication geometry (2×2 case)

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(\vec{V}_1 \vec{V}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\vec{V}_1 + c_2\vec{V}_2$

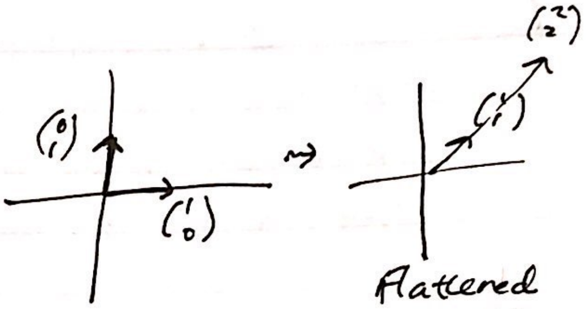


$0 \le c_1 \le 1$

$0 \le c_2 \le 1$

Collapse

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



Flattened parallelogram

$A\vec{x}$ only covers a line not a plane

Even in 3D, but especially for $n\times n$ matrices, dealing with whether we can solve $A\vec{x}=\vec{b}$ is serious.

Ex. 1

$$\begin{cases} x+4y-z=1 \\ 2y+z=3 \\ -2x+2y+2z=2 \end{cases}$$

① Augmented matrix $(A|\vec{b})$

pivot

$$\left(\begin{array}{ccc|c} \boxed{1} & 4 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ -2 & 2 & 2 & 2 \end{array}\right)$$

3x4 matrix

elementary row operations w/ goal of ~~eliminating things~~ creating zeros

② multiplying row 1 x2 + row 3

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & \boxed{2} & 1 & 3 \\ 0 & 10 & 0 & 4 \end{array}\right) \begin{array}{l} \leftarrow \text{same} \\ \leftarrow \text{same} \\ \leftarrow r_3=r_3+2r_1 \end{array}$$

③ now repeat on smaller matrix (dotted)

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & -5 & -11 \end{array}\right) r_3=r_3-5r_2$$

really efficient algorithm for machines..

row echelon form (REF)

Solve using back substitution

$$\frac{15}{5}-\frac{11}{5}=\frac{4}{5}$$

$$x+4y-z=1$$
$$2y+z=3$$
$$-5z=-11$$

$z=\frac{11}{5}$, $y=\frac{3-\frac{11}{5}}{2}=\cancel{\frac{4}{5}}\ \frac{2}{5}$.

$x=8/5$

Lecture 18: Gauss-Jordan Algorithm

3/18/19

- $A\vec{x}=\vec{b}$; $\vec{x}=A^{-1}\vec{b}$
- REF (Row eschelon form) and reduced REF
- non-invertible A, nullspace $A\vec{x}=\vec{0}$

Ex 1.

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ -2 & 2 & 2 & 2 \end{array}\right) \xrightarrow[\text{operations}]{\text{row}} \overset{\text{REF}}{\left(\begin{array}{ccc|c} \boxed{1} & 4 & -1 & 1 \\ 0 & \boxed{2} & 1 & 3 \\ 0 & 0 & \boxed{-5} & -11 \end{array}\right)}$$

backsubstitution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/5 \\ 2/5 \\ 11/5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & \boxed{1} & 11/5 \end{pmatrix} \quad \tilde{r}_3 = -\frac{1}{5} r_3 \quad \rightarrow \begin{pmatrix} 1 & 4 & 0 & * \\ 0 & 2 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \rightarrow \overset{\text{reduced REF}}{\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}}$$

$$\downarrow \begin{pmatrix} 8/5 \\ 2/5 \\ 11/5 \end{pmatrix}$$

Finding A^{-1} (PS 5)

$A\vec{x}_j = \vec{e}_j$, $j=1,2,3$ all at once.

Gauss-Jordan.

Augmented 3x6 $\quad \left(A \middle| \underbrace{\vec{e}_1, \vec{e}_2, \vec{e}_3}_{I}\right) = \left(A \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) = (A|I) \sim (I|\underbrace{\vec{x}_1|\vec{x}_2|\vec{x}_3}_{\overset{\uparrow}{B}\ 3\times 3})$

$A(\vec{x}_1|\vec{x}_2|\vec{x}_3) = (\vec{e}_1|\vec{e}_2|\vec{e}_3)$

$AB = I$
↑
right inverse

Definition: An $n \times n$ matrix A is called invertible if there is a matrix B such that $AB=BA=I$

Thm. The Gauss-Jordan algorithm produces $B=A^{-1}$

Ex 2

$$\begin{pmatrix} 1 & 4 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ -2 & -4 & 4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & -1 & 1 \\ 0 & \boxed{2} & 1 & 3 \\ 0 & 4 & 2 & 4 \end{pmatrix} \quad \tilde{r}_3 = r_3 + 2r_1$$

A is not invertible

$$\rightarrow \begin{pmatrix} 1 & 4 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & \textcircled{0} & -2 \end{pmatrix} \quad 0x + 0y + 0z = -2 \quad \text{no solution}$$

Ex. 3

$A\vec{x}=\vec{0}$ (homogeneous) i.e. $A\vec{x}=\vec{b}$ with $\vec{b}=\vec{0}$

$$\rightsquigarrow \left(\begin{array}{ccc|c} \boxed{1} & 4 & -1 & 0 \\ 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & \textcircled{0} & 0 \end{array}\right)$$

missing pivot

z is called a free variable

$$0x+0y+0z=0$$

$$2y+z=0 \quad y=-z/2$$

$$x+4y-z=0 \quad x=3z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3z \\ -z/2 \\ z \end{pmatrix}$$

Solutions form a 1 dim. of vectorspace

$$\left\{ c\begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix} \right\} = \underset{\text{null space of } A}{NS(A)}$$

$c \leftrightarrow z$

Ex. 4

$A\vec{x}=\vec{0}$ A is 3x5. 3 rows 5 columns,
3 equs. 5 unknowns

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \rightsquigarrow \begin{array}{ccccc} \boxed{3} & 1 & 1 & 1 & 1 \\ 0 & \boxed{2} & 3 & -1 & 0 \\ 0 & 0 & \textcircled{0} & \boxed{7} & \textcircled{4} \\ \boxed{x_1} & \boxed{x_2} & \textcircled{x_3} & \boxed{x_4} & \textcircled{x_5} \end{array} \rightarrow \begin{pmatrix} \boxed{1} & 0 & * & 0 & * \\ 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * \end{pmatrix}$$

REF

pivot variables $\boxed{x_4}$

free variables ○

Back substitution: try $(x_3,x_5)=(1,0) \rightsquigarrow \vec{v}_1$
& $(x_3,x_5)=(0,1) \rightsquigarrow \vec{w}$

$$NS(A)=\{x_3\vec{v}+x_5\vec{w}\}=\text{span}(\vec{v},\vec{w}) \quad \text{dim } 2$$

Scanned with CamScanner

Recitation

3/19/19

row-echelon form

$$\begin{pmatrix} \textcircled{1} & 4 & 2 & 1 & 3 \\ 0 & 0 & \textcircled{3} & 1 & 4 \\ 0 & 0 & 0 & 0 & \textcircled{2} \end{pmatrix}$$

the first nonzero entry occurs farther to the right

reduced row-echelon form

$$\begin{pmatrix} \textcircled{1} & 4 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{pmatrix}$$

- is in row-eschelon form ✓
- first non-zero entry has to be a 1 or a 0
- if row has first non-zero entry, then numbers above + below must be 0.

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

row operations

1. swap two rows → i.e. swap 1st + 3rd: $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$
2. add a multiple of a row to another row. → 2X row 2 + row 1 $\begin{pmatrix} 4 & 4 & 8 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$
3). Multiply a row by a scalar → i.e. row×2 $\begin{pmatrix} 2 & 2 & 4 \\ 3 & 3 & 6 \\ 1 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Matrices

$$\begin{cases} x + 3y = 11 \\ 2x + y + z = 6 \\ x + y + z = 4 \end{cases} \quad \Bigg| \quad \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 11 \\ 2 & 1 & 1 & 6 \\ 1 & 1 & 1 & 4 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 3 & 0 & 11 \\ 0 & -5 & 1 & -16 \\ 0 & -2 & 1 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 11 \\ 0 & -5 & 1 & -16 \\ 0 & 1 & -1/2 & 7/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 & 11 \\ 0 & 1 & -1/2 & 7/2 \\ 0 & -5 & 1 & -16 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 0 & 1 & -1/2 & 7/2 \\ 0 & 0 & -3/2 & 3/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 0 & 1 & -1/2 & 7/2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Matrices

things that can happen

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right)$$ ← no solution

pivots

$$\left(\begin{array}{ccc|c} \textcircled{1} & -1 & 0 & 2 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - y = 2$

$z = 3$

rows without pivots called free variables

as long as hold $\begin{cases} x = 2 + y \\ z = 3 \end{cases}$ infinitely many solutions

$$\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \textcircled{1} & -2 & 0 & 4 & 0 & -1 \\ 0 & 0 & \textcircled{1} & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 4 \end{pmatrix}$$

2 free variables.

$x_1 = -1 + 2x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 4 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -4 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - 2x_2 + 4x_4 = -1 \\ x_3 - 2x_4 = -1 \\ x_5 = 4 \end{cases}$$

$x_1 = -1 + 2x_2 - 4x_4$

$x_2 = x_2$

$x_3 = -1 + 2x_4$

$x_4 = x_4$

$x_5 = 4$

$$\rightarrow \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 4 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -4 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

3/20/19

Lecture 19: Solving $A\vec{x}=\vec{b}$ systematically

- $\vec{x}=\vec{x_p}+\vec{x_h}$; $(A|\vec{b})$
- nullspace, column space, dimension, rank
- determinants

Solve $A\vec{x}=\vec{b}$

one piece: $A\vec{x_p}=\vec{b}$

$\vec{x}_p$ particular solution

$$A(\vec{x}-\vec{x}_p) \underset{\substack{\uparrow \\ \text{distributive law}}}{=} A\vec{x}-A\vec{x_p} = \vec{b}-\vec{b}=\vec{0}$$

Hence $\vec{x}-\vec{x_p}=\vec{x_h}$ some homogeneous solution $A\vec{x_h}=\vec{0}$

general solution: $\boxed{\vec{x}=\vec{x_p}+\vec{x_h}}$

Examples 2 & 3 from last time $A\vec{x}=\vec{b}$

$$\left(\begin{array}{ccc|c}1&4&-1&b_1\\0&2&1&b_2\\-2&-4&4&b_3\\ \uparrow&\uparrow&\uparrow A&\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}\boxed{1}&4&-1&b_1\\0&\boxed{2}&1&b_2\\0&0&\textcircled{0}&b_3+2b_1-2b_2\\x&y&\textcircled{z}\text{ free}&\end{array}\right) \quad \boxed{\begin{array}{l}0=b_3+2b_1\\ \quad -2b_2\end{array}}$$

Case A: $b_3+2b_1-2b_2\neq 0$

No solution.

Case B: $b_3+2b_1-2b_2=0$ Always solutions $\rightarrow \vec{x}=\vec{x_p}+\vec{x_h}$

last equation

$0z=0$

Step 1: Solve $A\vec{x}=\vec{0}$ ($\vec{x_h}$ part)

We did this last time $b_1=b_2=b_3=0$

set free variable $z=1$. Now use back substitution.

$2y+1=0$ $y=-\frac{1}{2}$ $x+4(-\frac{1}{2})-1=0 \Rightarrow x=3$

$\boxed{\begin{pmatrix}3\\-1/2\\1\end{pmatrix} \text{ solves } A\vec{x}=\vec{0}}$

NS(A) = all $\vec{x}$ such that $A\vec{x}=0$

(nullspace) $= \text{span}\left\{\begin{pmatrix}3\\-1/2\\1\end{pmatrix}\right\}$ 1 dimensional vector space

$= \left\{c\begin{pmatrix}3\\-1/2\\1\end{pmatrix}\right\}$

Step2 $A\vec{x}=\vec{b}$ with $z=0$ (easiest)

$2y+1\cdot 0=b_2$ $y=b_2/2$

$x+4\left(\frac{b_2}{2}\right)-1\cdot 0=b_1$ $x=b_1-2b_2$

$\vec{X}_p=\begin{pmatrix} b_1-2b_2 \\ b_2/2 \\ 0 \end{pmatrix}$

only works when $b_3+2b_1-2b_2=0$ $\Rightarrow$ $\boxed{\vec{X}=\begin{pmatrix} b_1-2b_2 \\ b_2/2 \\ 0 \end{pmatrix}+c\begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}}$

Nullspace(A) = span $\left\{\begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}\right\}$ dim 1 ("nullity" of A) ✓

Column space

CS(A) = span of all columns ⇔ all $\vec{b}$ for which $A\vec{x}=\vec{b}$ has a solution

$=\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},\begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix},\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}\right\}$ ← redundant

CS(A) = span pivot columns which are always independent and hence a basis for the column space

$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},\begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix}\right\}=\text{all } \vec{b}$ Colums come from original matrix

Such that $b_3+2b_1-2b_2=0$ ← only two because $\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ is not a pivot column

2 dimensional plane

Defn rank(A) = dim(CS(A)) in our case rank=2

Theory "rank-nullity" rank+nullity = #columns

Proof: rank = # pivots, nullity = dim(NS(A)) = # free variables (non pivot columns)

Ex 4 from last time

non pivot columns

$$\underbrace{\begin{pmatrix} 0 & 2 & 3 & -1 & 0 \\ 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 7 & -14 \end{pmatrix}}_{A}\left|\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}\right. \xrightarrow{} \left(\begin{matrix} \boxed{3} & 1 & 1 & 1 & 1 \\ 0 & \boxed{2} & 3 & -1 & 0 \\ 0 & 0 & 0 & \boxed{7} & -14 \end{matrix}\right|\left.\begin{matrix} b_2 \\ b_1 \\ b_3 \end{matrix}\right)$$

Gauss elim.

$B = REF(A)$

row operations to REF

~~Dis~~ $NS(A) = NS(B)$ $\quad A\vec{x} = \vec{0} \Leftrightarrow B\vec{x} = \vec{0}$

2 free variable x_5 and x_3 $\leftarrow$ $\vec{v} = \begin{pmatrix} \cdot \\ \cdot \\ 1 \\ \cdot \\ 0 \end{pmatrix}$ $\leftarrow$ $(x_3, x_5) = (1, 0)$

$NS(A) = \text{span}\{\vec{v}, \vec{w}\}$ $\quad$ dim 2 = # free variable

$(x_3, x_5) = (0, 1)$

$\vec{w} = \begin{pmatrix} \cdot \\ \cdot \\ 0 \\ \cdot \\ 1 \end{pmatrix}$

0 and 1 used ~~found~~ for convenience

$CS(A) = $ span of pivot columns $\left\{ \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 7 \end{pmatrix} \right\}$

dim 3 # pivot variable

you must copy from A <u>not</u> row escheloun form

note $\underline{2+3=5}$

$$(B|\vec{b}) \xrightarrow[\text{jordan}]{} \left(\begin{matrix} \boxed{1} & 0 & -\frac{1}{6} & 0 & \frac{4}{3} \\ 0 & \boxed{1} & \frac{2}{3} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{matrix}\right|\left.\begin{matrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{matrix}\right)$$

$$\to \begin{pmatrix} 1 & 0 & -1/6 & 0 & 4/3 \\ 0 & 1 & 2/3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1/6 \\ -2/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1/6 & 0 & 4/3 \\ 0 & 1 & 2/3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -4/3 \\ +1 \\ 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \quad \end{pmatrix} \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \\ \beta_3 \\ 0 \end{pmatrix}}_{\vec{x}_p} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \vec{w}$$

$(x_3, x_5) = (0, 0)$

Lecture 20: Eigenvalues, ~~Eigenvalue~~ Eigenvectors, Diagonalization 3/22/19

- invertibility and determinants
- e-vals & e-vecs

<u>Theorem</u>

all are true or all are false

If A is a square matrix $n \times n$ then the following are equivalent:

1) A^{-1} exists (A is invertible)
2) $A\vec{x} = \vec{b}$ has <u>exactly</u> one solution for ~~every~~ every $\vec{b} \in \mathbb{R}^n$
3) A has n pivots
4) $NS(A) = \{0\}$ (nullity of $A = \dim NS(A) = 0$)
5) $CS(A) = \mathbb{R}^n$ (rank $A = \dim CS(A) = n$, full rank)
6) $\det(A) \neq 0$ (A is called non-singular)

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

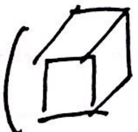
$n \times n$ $n!$ terms

warning: 10×10 ~ 5[illegible] raw ops (!!!) $10! \rightarrow 10^6 \ldots$

<u>properties of det</u>

$\det(A) = 0 \iff$ A is "singular"

$\iff$ A collapsed $\mathbb{R}^n$ to a lower dim - column space

A () has volume $|\det(A)|$

↑ unit cube

$\det A < 0$ means orientation reversing

$$A\left(\vec{x} : \begin{matrix} 0 \le x_1 \le 1 & 0 \le x_3 \le 1 \\ 0 \le x_2 \le 1 & \end{matrix}\right)$$

$\det(AB) = (\det A)(\det B) = \det(BA)$

Recall an eigenvalue λ of a (always square) matrix A is a number λ such that there exists a $\vec{v} \neq \vec{0}$ for which $\boxed{A\vec{v} = \lambda\vec{v}}$

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} = \lambda I\vec{v}$ I = identity matrix add I for $\Leftrightarrow$ parallel structure $(A-\lambda I)\vec{v} = \vec{0}$

$\vec{v}$ is in $NS(A-\lambda I)$ $\Leftrightarrow$ $A-\lambda I$ is singular

In particular $\det(A-\lambda I) = 0$ ...

$\uparrow$

degree n poly in λ

Example 1

$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow$ cubic eqn for λ

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda(\lambda^2-1) - (-\lambda-1) + (1+\lambda)$$

$$= -\lambda^3 + 3\lambda + 2 = 0 = -(\lambda+1)^2(\lambda-2)$$

$\boxed{\lambda = -1}$ is a double root $\boxed{\lambda = 2}$ another root

Eigenvectors

$\lambda_1 = -1$

$\lambda_2 = -1$

$\lambda_3 = 2$

$$\begin{pmatrix} -(-1) & 1 & 1 \\ 1 & -(-1) & 1 \\ 1 & 1 & -(-1) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2 dim null space

$\binom{1}{1}$, $\binom{1}{-1}$ these are perpendicular

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

matrix of eigenvectors

$\downarrow$

S D diagonal

$A\vec{v}_j = \lambda_j \vec{v}_j$

$$A(\vec{v}_1 | \vec{v}_2 | \vec{v}_3) = (\lambda_1 \vec{v}_1 | \lambda_2 \vec{v}_2 | \lambda_3 \vec{v}_3) = (v_1|v_2\, v_3)\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}.$$

$\boxed{AS = SD}$ multiply by S^{-1}

Definition An eigenvalue is called complete if it has as many eigenvectors as its multiplicity

eigenspace of λ has dimension = multiplicity of λ

When the eigenvalues are complete,
S is invertible $\boxed{A = SDS^{-1}}$ $(AS)S^{-1} = SDS^{-1}$

Lecture 21: Diagonalization   Exam 2 Review   4/1/19

exam: 7-9 pm Room 2-255
A-N Walker Memorial Top floor
O-Z 26-100

Recall: A nxn matrix

If $A\vec{v}=\lambda\vec{v}$, then $\vec{x}(t)=e^{\lambda t}\vec{v}$ solves $\dot{\vec{x}}=A\vec{x}$

A matrix A is called <u>complete</u> if it has a basis of eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ such that $A\vec{v}_j=\lambda_j\vec{v}_j$ (lin. independence)

In this situation $S=(\vec{v}_1|\vec{v}_2|\dots|\vec{v}_n)$ is fullrank so that S^{-1} exists.

$$\boxed{A\vec{v}_j=\lambda\vec{v}_j \quad j=1,\dots,n} \Leftrightarrow \underbrace{A=SDS^{-1}}_{\text{diagonalization}}, \quad D=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

<u>PROOF</u> ($\Rightarrow$) $A\vec{v}_j=\lambda_j\vec{v}_j \Rightarrow AS=SD \Rightarrow A=SDS^{-1}$

($\Leftarrow$) If $A=SDS^{-1}$ $\vec{e}_j=\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}\leftarrow j$th

$A\vec{v}_j=SDS^{-1}\vec{v}_j=SDe_j=S\lambda_j\vec{e}_j=\lambda_j\vec{v}_j$

$\boxed{A\vec{v}_j=\lambda_j\vec{v}_j}$

characteristic polynomial
$P(\lambda)=\det(\lambda I-A)\leftarrow$ degree n
$=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$ may have repeats

Example from before vacation:

$A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$ $\lambda_1=2$, $\vec{v}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ | $\lambda_2=-1$ $\vec{v}_2=\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\lambda_3=-1$ $\vec{v}_3=\begin{pmatrix}1\\1\\2\end{pmatrix}$ (2 dimensional eigenspace)

$P(\lambda)=(\lambda-2)(\lambda-(-1))^2=(\lambda-2)(\lambda^2+2\lambda+1)=\lambda^3-3\lambda-2$

$\boxed{\mathrm{trace}(A)=\lambda_1+\dots+\lambda_n}$

$\boxed{\det(A)=\lambda_1\lambda_2\dots\lambda_n}$

<u>Theorem 1</u> If the roots of the charteristic polynomial are distinct, then the matrix is complete

<u>Theorem 2</u> If the matrix is <u>real</u> and symmetric ($A^T=A$), then there is an <u>orthogonal</u> basis of eigenvectors (and the eigenvalues are real)

$\begin{pmatrix}a&d&e\\d&b&f\\e&f&c\end{pmatrix}^T=$(itself) $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}$ are perpendicular to each other.

Ex. complex eigenvalues (and eigenvectors) do occur.
But the 2x2 case illustrates everything we need.

(use to describe systems that have oscillatory behavior)

PS 6 rotations in 3D

$A = S\begin{pmatrix} 1 & 0 & 0 \\ 0 & \boxed{\text{rotatation}} & \\ 0 & \boxed{\text{matrix}} & \end{pmatrix} S^{-1}$

What's on the test:

- complex replacement, gain, resonance
- 2x2 systems, eigenvalues, eigenvectors, phase portraits
- solving $A\vec{x}=\vec{b}$ by elimination $(\vec{x}=\vec{x}_p+\vec{x}_h)$

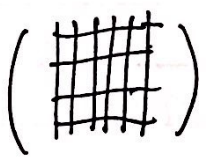
1. ERF, ERF' are written on Exam 2
2. Memorize the trace-determinant plane & terminology
   = names of structurally stable types
3. Need basis, CS(A), basis of NS(A), rank, nullity, dimension counting

Q. Limits of dimensions of space (related rank-nullity ~~too~~ theorem)

$$\underbrace{\text{rank}}_{\text{dim. of column space}} + \underbrace{\text{nullity}}_{\text{dim. of nullspace}} = \#\text{ columns}$$

Example If A is mxn matrix and $A\vec{x}=\vec{b}$ has solutions for some but not all nonzero $\vec{b}$, then what are the limitations on rank A?

m=4
4 rows



n=6
6 columns

$\boxed{m\times n}$

rank A + nullity A = 8

m=5, n=8

~~Some solution~~

$\dim(CS(A)) < 5$ columns. $\mathbb{R}^5$

$0 < \text{rank } A < 5 \implies \text{nullity}(A) = \dim NS(A)$ is ~~strictly~~ between 8-4 and 8-1

4 and 7

Recitation 14

4/4/19

Characteristic polynomial:

↳ $\det(A-\lambda I)$

$$\begin{pmatrix} 1-\lambda & -1 & 4 \\ -1 & 4-\lambda & -1 \\ 4 & -1 & 1-\lambda \end{pmatrix} \quad (1-\lambda)\det\begin{pmatrix} 4-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} - (-1)\det\begin{pmatrix} -1 & 4 \\ -1 & 1-\lambda \end{pmatrix}$$

$$+4\det\begin{pmatrix} -1 & 4 \\ 4-\lambda & -1 \end{pmatrix}$$

$= (1-\lambda)((4-\lambda)(1-2)-1) + ($… I'm not going to write it out)

$= (\lambda^3-6\lambda^2-9\lambda+54) = (\lambda-6)(\lambda-3)(\lambda+3)$

$\lambda = 6, 3, -3$

Find the eigenvector $\Rightarrow NS(A-3I)$

$$\lambda=3 \quad A-3I = \begin{pmatrix} -2 & -1 & 4 \\ -1 & 1 & -1 \\ 4 & -1 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$A = SDS^{-1}$ ← inverse of eigenvectors

↑ eigenvectors → diagonal of eigenvalues

i.e.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}^{-1}$$

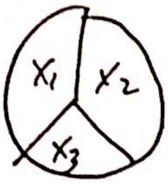
↑ eigenvectors: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, eigenvalues: 3, 6, −3

A eigenvalue λ

$NS(A-\lambda I)$ can be larger than 1 dim.

DE Lecture 22: Decoupling, Fundamental Solutions 4/5/19

Example:



X_k salinity

$\dot{x}_1 = a(x_2 - x_1) + a(x_3 - x_1)$

$= a(-2x_1 + x_2 + x_3)$

$\boxed{a=1}$ $\dot{x}_1 = (-2\ 1\ 1)\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

By symmetry:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \qquad A = \begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix} \qquad \dot{\vec{x}} = A\vec{x}$$

eigenvalues + eigenvectors $\lambda_1, \lambda_2, \lambda_3$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

general solution

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3$$

$\lambda_1 = 0$ $\vec{v}_1 = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$ $\lambda_2 = -3$ $\vec{v}_2 = \begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$ $\lambda_3 = -3$ $\vec{V}_3 = \begin{pmatrix}1\\ 1\\ -2\end{pmatrix}$

$A = SDS^{-1}$

$S = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ $\vec{x}(t) = y_1(t)\vec{v}_1 + y_2(t)\vec{v}_2 + y_3(t)\vec{v}_3$ ($\vec{v}_1, \vec{v}_2, \vec{v}_3$ is basis)

$D = \begin{pmatrix}\boxed{0} & 0 & 0\\ 0 & \boxed{-3} & 0\\ 0 & 0 & \boxed{-3}\end{pmatrix}$

$\dot{\vec{x}} = A\vec{x} \Leftrightarrow$

$\sum_{k=1}^{3} \boxed{\dot{y}_k}\vec{v}_k = A\sum y_k \vec{v}_k = \sum \boxed{y_k \lambda_k} v_k$

$\Rightarrow$ independence $\boxed{\dot{y}_k = \lambda_k y_k}$ $k = 1, 2, 3$

$\vec{x} = (\vec{v}_1\ \vec{v}_2\ v_3)\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$ $\vec{x} = S\vec{y}$ $\boxed{S^{-1}\vec{x} = \vec{y}}$

$S^{-1} = \begin{pmatrix}v_1^T/3\\ v_2^T/2\\ v_3^T/6\end{pmatrix}$ $S^{-1}S = I$

$V_1^T V_1 = \|v_1\|^2 = 3$ $V_3^T V_3 = \|V_3\|^2 = 6$

$V_2^T V_2 = \|V_2\|^2 = 2$

$$\boxed{\begin{array}{c} V_j V_K^T = 0 \\ \text{if } j \neq K \end{array}}$$

$$y_1 = \left(V_1^T/3\right)\vec{x} = \frac{1}{3}(x_1 + x_2 + x_3) \quad \vec{V_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$y_2 = \left(V_2^T/2\right)\vec{x} = \frac{1}{2}(x_1 - x_2)$$

$$y_3 = \left(V_3^T/6\right)\vec{x} = \frac{1}{6}(x_1 + x_2 - 2x_3)$$

these quantities are $\boxed{\text{decoupled}}$.

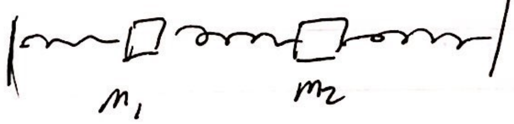
$$\dot{y}_1 = 0 \quad \boxed{y_1(t) = y(0)e^{0t}}$$

$$\dot{y}_2 = -3y_2 \quad y_2(t) = y_2(0)e^{-3t}$$

Example 2 P56

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{R(t)} \\ 0 & & \end{array}\right)$$

Example 3 in MITX notes



4x4 $\left(\begin{array}{c|c} R(\omega t) & \\ \hline & R(\sqrt{3}\omega t) \end{array}\right)$

Fundamental Solutions

A fundamental solution of $\dot{\vec{x}} = A\vec{x}$ is an $n \times n$ matrix whose columns are independent solutions

$$X(t) = (\vec{x}_1(t), \ldots \vec{x}_n(t))$$

For example

$$\underline{X}(t) = \left(e^{\lambda_1 t}\vec{V_1} \,\middle|\, e^{\lambda_2 t}\vec{V_2} \,\middle|\, e^{\lambda_3 t}\vec{V_3}\right) = \begin{pmatrix} 1 & e^{-3t} & e^{-3t} \\ 1 & -e^{-3t} & e^{-3t} \\ 1 & 0 & -2e^{-3t} \end{pmatrix}$$

3x3 matrix

Defining properties of fundamental solution(s)

1) determinant $\underline{X}(0) \neq 0$

2) ~~matrix~~ $\dot{\underline{X}} = A\underline{X}$ matrix differential equation

why ② ? $\dot{\underline{X}} = (\dot{\vec{x}}_1 | \dot{\vec{x}}_2 | \dot{\vec{x}}_3) = (A\vec{x}_1 | A\vec{x}_2 | A\vec{x}_3) = A(\vec{x}_1 | \vec{x}_2 | \vec{x}_3) = A\underline{X}$

1) allows us to solve the initial value problem

$$\vec{x}(t) = C_1\vec{x}_1(t) + \cdots + C_n\vec{x}_n(t) = \underline{X}(t)\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = \underline{X}(t)\vec{C} \quad \text{general solution}$$

$$\vec{x}(0) = \underline{X}(0)\vec{C} \iff \left(\vec{C} = \underline{X}(0)^{-1}\vec{x}(0)\right) \quad \text{inverse exists because the determinant}(\underline{X}(0)) \neq 0$$

Theorem If $X(t)$ is a fundamental solution and $\det(X(0)\neq 0)$
then $\det(\overline{\underline{X}}(t)\neq 0)$ for all t.

Prof $W(t)=\det(\overline{\underline{X}}(t))$ (wranskian)

$\dot{W}(t) = \text{trace}(A)W(t)$ $\quad W(t)=W(0)e^{tr(A)t} \neq 0$

even for $A=A(t)$, the theorem works

$$\det(\overline{\underline{X}}(t)) = e^{-3t}e^{-3t}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = e^{-6t}\det(\overline{\underline{X}}(0))$$

$$A=\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad \text{trace}(A) = -6$$

Lecture 23: 4/8/19

Exam 2 median: 83 $A \geq 87$ $B \geq 73$ $C \geq 60$

Fundamental Matrix : $\dot{\mathbf{X}} = A\mathbf{X}$ ($\det \mathbf{X}(0) \neq 0$)

Variation of Parameters: $\dot{\vec{x}} = A\vec{x} + \vec{r}(t)$ (inhomogeneous prob)

exponential matrix : e^{At}

Review : A fundamental matrix $\mathbf{X}(t)$ in a $n \times n$ matrix satisfying

$\dot{\mathbf{X}} = A\mathbf{X}$, $\det(\mathbf{X}(0)) \neq 0$

Thm: $\det(\mathbf{X}(t)) \neq 0$ for all t.

homogeneous problem $\vec{x} - A\vec{x} = 0$

$$\dot{\mathbf{X}} - A\mathbf{X} = 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\vec{x_h} = c_1\vec{x_1} + c_2\vec{x_2}$ 2 dim if

$= (\vec{x}_1 | \vec{x}_2) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{X}\vec{c}$

Inhomogeneous problem

$\dot{\vec{x}} - A\vec{x} = \vec{r}(t)$ $(\dot{\vec{x}} = A\vec{x} + \vec{r}(t))$

Method of $\boxed{\text{Variation of parameters}}$ $\boxed{\text{TRY: } \mathbf{X}\vec{u}}$ $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$c_1, c_2 \rightsquigarrow u_1(t), u_2(t)$

1 dimensional case

$\dot{x} - ax = r(t)$ | $\overset{\text{TRY}}{x(t) = u(t)e^{at}} \rightarrow x(t) = e^{at}\left(\int e^{-at} r(t)\,dt + c\right)$

$x_h = ce^{at}$ | $x(t) = x_p(t) + x_h(t)$ $x_h(t) = ce^{at}$

Plug in $\vec{x} = \mathbf{X}\vec{u}$ into $\dot{\vec{x}} - A\vec{x} = \vec{r}(t)$

$(\mathbf{X}u)^{\cdot} - A(\mathbf{X}u) = r(t)$

$\dot{\mathbf{X}}\vec{u} + \mathbf{X}\dot{\vec{u}} - A(\mathbf{X}\vec{u}) = \vec{r}(t)$ $(AB)C = A(BC)$

$\cancel{(A\mathbf{X})\vec{u}} + \mathbf{X}\dot{\vec{u}} - \cancel{A(\mathbf{X}\vec{u})} = \vec{r}(t)$ associative law, shift the parenthesis.

$\Rightarrow \mathbf{X}\dot{\vec{u}} = \vec{r}$ $\boxed{\dot{\vec{u}} = \mathbf{X}(t)^{-1}\vec{r}(t)}$ ($\mathbf{X}(t)$ nonsingular)

$$\vec{u} = \int \underline{X}(t)^{-1}\vec{r}(t)\,dt$$

$$\vec{x} = \underline{X}(t)\left(\int \underline{X}(t)^{-1}\vec{r}(t)\,dt + \vec{c}\right)$$

$$\vec{x}_h = \underline{X}(t)\vec{c}$$

Exponential:

definition e^{At} is the $n \times n$ matrix $E(t)$ satisfying $\dot{E} = AE$ and $E(0) = I$

$$\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$$

Analogy

$\dot{y} = ay \quad y(0) = 1$

$y(t) = e^{at}$

$\dot{z} = iz \quad z(0) = 1$

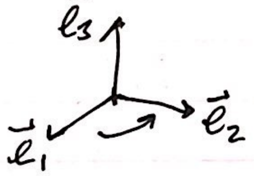
$z(t) = e^{it} \quad i^2 = -1$

PS6 $i \leftrightarrow$ rotation by 90°

$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow J^2 = -I$

e^{Jt} rotation by t radian

Rotations in 3dims.



$J_3 = e^{(\pi/2)J_3}$

$i = e^{(\pi/2)i}$

$J_1 J_3 \neq J_3 J_1$

$e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$

but

$e^{A_1+A_2} \neq e^{A_1}e^{A_2}$

nearly always false

$e^{A_1}e^{A_2} \neq e^{A_2}e^{A_1}$

addition is commutative but matrix multiplication is not.

Example 1

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} \lambda_1 e^{\lambda_1 t} & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_1 \end{pmatrix} \to \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$$= De^{Dt}$$

Example 2

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

$\lambda_1 = 4 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -2 \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$S = (v_1 | v_2) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$(e^{\lambda_1 t}\vec{v}_1 | e^{\lambda_2 t}v_2)$

$$\underline{X}(t) = \begin{pmatrix} e^{4t} & e^{-2t} \\ -e^{4t} & e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{pmatrix} = Se^{Dt} \quad \text{where } D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

What about e^{At}?

$\underline{X}(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ not $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

if we start with one fundamental solution, we can always generate the rest.

$F(t) = \underline{X}(t)\vec{C}$ any constant matrix C

$$\dot{F} = \dot{\underline{X}}C + \underline{X}\cancel{\dot{C}} = (A\underline{X})C = A(\underline{X}C) = AF$$

$$\underline{\dot{F} = AF}$$

Choose $C = X(0)^{-1}$

$e^{At} = \underline{X}(t)\underline{X}(0)^{-1}$ and $\underline{X}(0)\underline{X}(0)^{-1} = I$ ✓

In our example, $\underline{X}(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = S$

$\underline{X}(0)^{-1} = S^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$$e^{At} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \underline{Se^{tD}S^{-1}}$$

Decoupling

"if there are 40 things to look at, we can only look at one"

$$S^{-1}\vec{r} = \begin{pmatrix} (1/2)r_1 - (\frac{1}{2})r_2 \end{pmatrix}$$

input $\frac{r_1 - r_2}{2} \rightsquigarrow \frac{x_1 - x_2}{2}$ particular response without looking at all other components

another way of defining the exponential (will help on problem set)

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

n×n matrices

one of the ways we deal with the exponential matrix

Lecture 24:

4/10/19

Exponentials: $e^{At} = I + At + (At)^2/2! + \dots = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$

numerical methods (euler's method)

Ex. 1 $D = \begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}$ $D^2 = \begin{pmatrix}\lambda_1^2 & 0 \\ 0 & \lambda_2^2\end{pmatrix}$ $D^n = \begin{pmatrix}\lambda_1^n & 0 \\ 0 & \lambda_2^n\end{pmatrix}$

$$e^{Dt} = \sum \frac{(tD)^n}{n!} = \sum D^n \frac{t^n}{n!} = \sum \begin{pmatrix}\lambda_1^n & 0 \\ 0 & \lambda_2^n\end{pmatrix} \frac{t^n}{n!} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix}$$

Ex. 2. $A = \begin{pmatrix}1 & -3 \\ -3 & 1\end{pmatrix} = SDS^{-1} = \begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}4 & 0 \\ 0 & -2\end{pmatrix}\begin{pmatrix}\cancel{1/2}\ 1/2 & -1/2 \\ 1/2 & 1/2\end{pmatrix}$

$A^2 = SD\cancel{S^{-1}}\cancel{S}DS^{-1} = SD^2S^{-1}$ $A^n = SD^nS^{-1}$

$$e^{At} = \sum \frac{(At)^n}{n!} = \sum SD^nS^{-1}\frac{t^n}{n!} = S\left(\sum \frac{D^n t^n}{n!}\right)S^{-1}$$

$= Se^{Dt}S^{-1}$

$$e^{At} = \begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}e^{4t} & 0 \\ 0 & e^{-2t}\end{pmatrix}\begin{pmatrix}1/2 & -1/2 \\ 1/2 & 1/2\end{pmatrix}$$

don't multiply unless you have to

Ex. 3

$A = \begin{pmatrix}2 & 1 \\ -1 & 4\end{pmatrix}$ $\lambda^2 - 6\lambda + 9 = 0$ $(\lambda - 3)^2 = 0$ repeated root

$A = 3I + N$

$A - 3I = \begin{pmatrix}-1 & 1 \\ -1 & 1\end{pmatrix} = N$ $N^2 = \begin{pmatrix}-1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1 \\ -1 & 1\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}}_{\text{nilpotent}}$

$$e^{At} = e^{(3I+N)t} \underset{\uparrow}{=} e^{3It} \cdot e^{Nt} = \begin{pmatrix}e^{3t} & 0 \\ 0 & e^{3t}\end{pmatrix}\left(\qquad \right)$$

3I and N commute

$e^{Nt} = I + Nt + \cancel{\frac{(Nt)^2}{2!}} + \dots$

0 $+ 0 + 0 \dots$

$\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix}-t & t \\ -t & t\end{pmatrix}$

Solution should be built from e^{3t} and te^{3t}

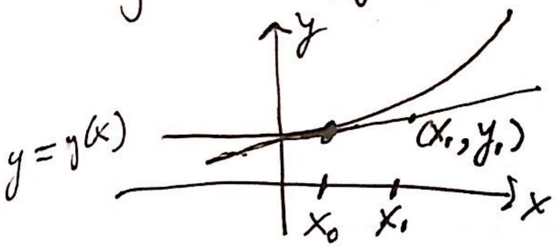
$\vec{v}_1 e^{3t} + \vec{v}_2 te^{3t}$ on solution is 1st column of e^{At}

$$\begin{pmatrix}(1-t)e^{3t} \\ -te^{3t}\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}e^{3t} - \begin{pmatrix}1 \\ 1\end{pmatrix}te^{3t}$$

Numerical Methods (Euler's Method)

Euler's Method (he just wants you to memorize it...)

$y'(x) = f(x, y(x))$ ← could be nonlinear



$y(x_0) = y_0$ initial condition

$x_1 - x_0 = h$ step size

$\frac{y_1 - y_0}{x_1 - x_0}$ = slope at x_0 $= f(x_0, y_0)$

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Ex on mathlet $f(x,y) = y^2 - x$

$h=1$ orange | $x_0 = 0$, $y_0 = 0$ | $h = 0.125 = 1/8$

$y' = y^2 - x$

Goal $y(x)$ should resemble the sequence $y_0, y_1, y_2 \dots$

$\underset{\text{approx.}}{y_n} - \underset{\text{exact}}{y(x_0 + nh)} = E$ (error) $|E| \leq C_h$ $(0 \leq nh \leq 3)$

(系 depends on $|f| + |\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}|$)

Implementation by hand

$h = 1/10$, $x_0 = 1$, $y_0 = 1$

Make a table

n	x_n	y_n	$f(x_n, y_n)$	$hf(x_n, y_n)$
0	1	1	$1^2 - 1 = 0$	0
1	1.1	1+0	$1^2 - 1.1 = -1$	-0.01
2	1.2	.99	$.99^2 - 1.2$	$\frac{1}{10}((.99)^2 - 1.2)$

Reliability

- self consistency. No crossing
- convergence as $h \to 0$
  - stability (boring)

Recitation

Decoupling Method

4/11/19

Inhomogeneous differential equation

$$\dot{x} - 2x = \cos(3t)$$

$$\vec{x}\begin{pmatrix} x \\ \vdots \\ x_n \end{pmatrix} \qquad \dot{\vec{x}} = \underset{\substack{\uparrow \\ n\times n \text{ matrix}}}{A}\vec{x} + \vec{g}(t)$$

Method 1: Decoupling

$$A = SDS^{-1}$$

$$\dot{\vec{x}} = A\vec{x} + \vec{g} = SDS^{-1}\vec{x} + \vec{g}$$

$$\downarrow S^{-1}$$

$$S^{-1}\dot{\vec{x}} = DS^{-1}\vec{x} + S^{-1}\vec{g}$$

$$S^{-1}\vec{x} = \vec{y} \qquad S^{-1}\vec{g} = \vec{p} \qquad \dot{\vec{y}} = D\vec{y} + \vec{p} \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{pmatrix}$$

$$\begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \vdots \\ \lambda_n y_n \end{pmatrix} + \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \qquad D\vec{y} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \vdots \\ \lambda_n y_n \end{pmatrix}$$

Get single variable equations

$$\underset{\substack{\uparrow \\ \text{solve using method from beginning of semester.}}}{\dot{y}_i} = \lambda_i y_i + p_i$$

Example

$$\begin{cases} \dot{x}_1 = \underbrace{-16x_1 + 6x_2}_{} + \boxed{t^2} \\ \dot{x}_2 = \underbrace{-45x_1 + 17x_2}_{\text{homogeneous part}} + \boxed{1+t} \end{cases} \text{ inhomogeneous}$$

$$\dot{\vec{x}} = \begin{pmatrix} -16 & 6 \\ -45 & 17 \end{pmatrix}\vec{x} + \overset{g(t)}{\begin{pmatrix} t^2 \\ 1+t \end{pmatrix}}$$

1) write $A = SDS^{-1}$

eigenvalues + vectors

$$2 \rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$\dot{\vec{x}} = SDS^{-1}\vec{x}+q$ $\quad S^{-1}\dot{\vec{x}} = DS^{-1}\vec{x} + S^{-1}q$ $\quad \vec{y} = S^{-1}\vec{x}$

$\vec{y} = D\vec{y} + S^{-1}q$

$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} t^{2} \\ 1+t \end{pmatrix} = \begin{pmatrix} 5t^{2}+2t+2 \\ 3t^{2}-t-1 \end{pmatrix}$$

$$\dot{\vec{y}} = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\vec{y}} + \begin{pmatrix} -5t^{2}+2t+2 \\ 3t^{2}-t-1 \end{pmatrix}$$

$\uparrow$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} \qquad \begin{pmatrix} 2y_1 \\ -y_2 \end{pmatrix}$$

$\dot{y}_1 = 2y_1 - 5t^{2} + 2t + 2$

$\dot{y}_2 = -y_2 - 3t^{2} - t - 1$

$\dot{y}_1 - 2y_1 = -5t^{2} + 2t + 2$

guess a solution to be of the form $at^{2}+bt+c$

$\dot{(at^{2}+bt+c)} - 2(at^{2}+bt+c) = -5t^{2}+2t+2$

$2at+b-2at^{2}-2bt-2c = -5t^{2}+2t+2$

$-2at^{2}+(2a-2b)t+(b-2c) = -5t^{2}+2t+2$

$-2a=5 \quad 2a-2b=2 \quad b-2c=2$

$y_1 = \frac{5}{2}t^{2} + \frac{3}{2}t - \frac{1}{4}$

$y_2 = 3t^{2}-t-1$

$$\vec{x} = S\vec{y} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} \frac{5}{2}\cancel{t}^{2}\frac{3}{2}t^{2}-\frac{1}{2} \\ 3t^{2}-t-1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{2}t^{2}-\frac{t}{2}-\frac{9}{4} \\ \frac{45}{2}t^{2}-\frac{t}{2}-2 \end{pmatrix}$$

Scanned with CamScanner

Method 2: Variation of parameters

$\dot{\vec{x}} = A\vec{x} + \vec{g}(t)$ $\qquad \underset{\text{homogeneous solution}}{\dot{\vec{X}} = A\vec{X}}$ $\qquad \dot{\vec{X}}_1 = A\vec{X}_n$ $\qquad X(t)$ fundamental solution.

guess that $\vec{x} = X(t)\vec{u}$

$\dot{\vec{x}} = A\vec{x} + \vec{g}$ $\qquad (X\dot{\vec{u}}) = AX\vec{u}$

$$\dot{X}\vec{u} + X\dot{\vec{u}} = AX\vec{u} + \vec{g}$$

$$X\dot{\vec{u}} + \underbrace{\cancel{(\dot{X} - AX)}}_{0}\vec{u} = \vec{g}$$

$$X\dot{\vec{u}} = \vec{g} \Rightarrow \dot{\vec{u}} = X^{-1}\vec{g}$$

Solve for $\vec{u}$

then

$$\vec{x} = X(t)\vec{u}$$

$$\dot{\vec{X}} = \begin{pmatrix} -16 & 6 \\ -45 & 17 \end{pmatrix}\vec{x} + \begin{pmatrix} e^{t} \\ e^{-t} \end{pmatrix} \rightarrow \begin{cases} \dot{x}_1 = -16x_1 + 6x_2 + e^{t} \\ \dot{x}_2 = -45x_1 + 17x_2 + e^{-t} \end{cases}$$

Find fundamental solution to

$\vec{X}_0 = A\vec{X}_0$

Eigenvalues / vectors

$2 \rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad -1 \rightarrow \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad e^{-t}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$X(t) = \begin{pmatrix} e^{2t} & 2e^{-t} \\ 3e^{2t} & 5e^{-t} \end{pmatrix}$$

$$X(t) = \begin{pmatrix} e^{2t} & 2e^{-t} \\ 3e^{2t} & 5e^{-t} \end{pmatrix} \qquad X(t)^{-1} = \frac{1}{\underbrace{ab-bc}_{-e^{t}}}\begin{pmatrix} 5e^{-t} & -2e^{-t} \\ -3e^{2t} & e^{2t} \end{pmatrix}$$

$$X(t)^{-1} = \begin{pmatrix} -5e^{-2t} & 2e^{2t} \\ 3e^{t} & -e^{t} \end{pmatrix}$$

$$X(t)^{-1}\vec{q} = \begin{pmatrix} -5e^{-2t} & 2e^{-2t} \\ 3e^{t} & -e^{t} \end{pmatrix}\begin{pmatrix} e^{t} \\ e^{-t} \end{pmatrix}$$

$$X^{-1}q = \begin{pmatrix} -5e^{-t} + 2e^{-3t} \\ 3e^{2t} + e^{-2t} \end{pmatrix} = \dot{u}$$

$$\dot{u}_1 = -5e^{-t} + 2e^{-3t} \qquad u_1 = 5e^{-t} - \frac{2}{3}e^{-3t} \text{ integrate}$$

$$\dot{u}_2 = 3e^{2t} + e^{-2t} \qquad u_2 = \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t}$$

$$\vec{X} = X(t)\vec{u} = \begin{pmatrix} e^{2t} & 2e^{-t} \\ 3e^{2t} & 5e^{-t} \end{pmatrix}\begin{pmatrix} 5e^{-t} - \frac{2}{3}e^{-3t} \\ \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 5e^{t} - \frac{2}{3}e^{-t} + 10e^{-2t} - \frac{4}{3}e^{-4t} \\ \cancel{15e^{t} \; 2e^{-t} + \frac{1}{2}e^{4t}} \\ \frac{25}{2}e^{t} - 2e^{-t} + \frac{15}{2}e^{t} - \frac{5}{2}e^{-3t} \end{pmatrix}$$

↑ final answer

from handout, do parts (a, d, e, f, g)

Lecture 25: Eigenvalues & Eigenvectors in disordered media

4/12/19

This is just him talking about his research...

$$\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \quad \dot{\vec{y}} = A\vec{y} \qquad A = \begin{pmatrix} -2 & 1 & & \\ \ddots & \ddots & \ddots & \\ & 1 & -2 & 1 \\ & & \ddots & -2 \end{pmatrix}$$

$$\vec{y}(t) = e^{At}\vec{y}(0)$$

Vibrations $\ddot{\vec{u}} = A\vec{u}$

$$\vec{u}(t) = \cos(At)\vec{u}(0) \quad \sin(At)$$

shrödinger equation $\dot{\vec{z}} = iA\vec{z}$ $\quad \vec{z}(t) = e^{iAt}\boxed{\vec{z}(0)}$

Biggest unsolved problem in condensed matter physics

shape of solutions... (?)

$$A \longleftrightarrow \frac{d^2}{dx^2} \qquad \frac{d^2}{dx^2}\cos(kx) = -k^2\cos(kx)$$

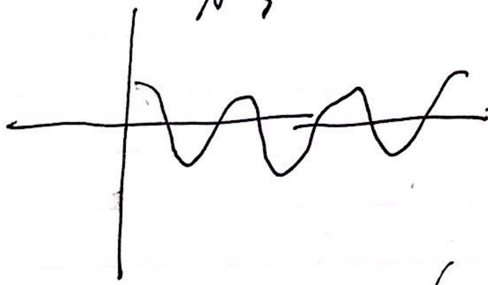
$\cos(kx)$ eigenfunction

$-k^2$ its eigenvalue

Shape:

$y = y(x) \quad 0 \leq x \leq 1$

$x = \frac{n}{N}, \; n = 0, \ldots, N$



Anderson model

$$\underbrace{-A + V}_{\text{disorder}}$$

$-\frac{d^2}{dx^2} + V(x) \leftarrow$ potential

$-A$

$$V(n) = \begin{cases} V^* & \frac{1}{10p} \\ 0 & \frac{9}{10p} \end{cases}$$

$$V = \begin{pmatrix} V(1) & & 0 \\ & \ddots & \\ 0 & & V(N) \end{pmatrix} \leftarrow \text{diagonal}$$

$V^* = 1/10$

What do the eigenvectors look like?

eigenfunctions localize

at certain energy levels, the eigenvalues / eigenfunctions are concentrated

low energies – highly localized
higher energies "extended" spread out

transition is called the **mobility edge.**

edge between when semiconductor is insulating or conducting

open question for 60 yrs

Quantum mechanics: particles are eigenvectors
We (everyone in physical space) are superposition of eigenvectors, evolving according to Schrödinger's equations.

→ we don't know initial condition
this is what discussions about cause & effects are about

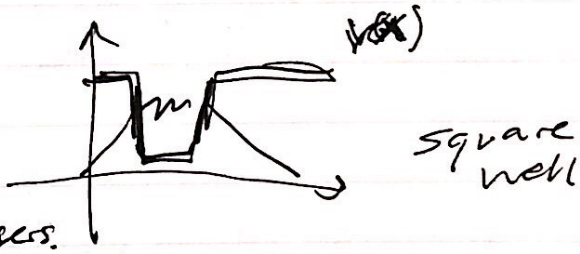
$\vec{Z}(t) = e^{iAt}\vec{Z}(0)$

$p_n = |Z_n(t)|^2$

prob. that the particle is at site n

$p_1 + .. + p_N = 1$

Example $-\frac{d^2}{dx^2} + V(x)$



bose einstein condensate
collect particles using lasers.
cools stuff down to nano kelvin.

$$\left(-\frac{d^2}{dx^2} + \overset{\| 0}{V(x)}\right)\cos(kx) = k^2\cos(kx)$$ Inside

$$\left(-\frac{d^2}{dx^2} + V^*\right)X(x) = +k^2 X(x)$$

$$-\frac{d\psi^2}{dx^2} = (k^2 - V^*)X(x)$$

|x| outside the well

$\psi(x) \sim e^{-(\sqrt{V^*-k^2})|x|}$

Uniform pressure problem

$$(-A+V)\vec{u} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \vec{u} = \begin{pmatrix} u(1) \\ \vdots \\ u(n) \end{pmatrix}$$

$V\vec{u} = 1$

$V(x) \Leftrightarrow \frac{1}{u(x)}$ have the same units.

$v(1), V(N)$
cliffs & waterways $\rightarrow$ $1/u(1)$, $1/u(N)$
Hills & Valleys

$$\begin{pmatrix} 1/u(1) & & \\ & \ddots & \\ & & 1/u(N) \end{pmatrix} (-A+V) \begin{pmatrix} u(1) & & 0 \\ & \ddots & \\ 0 & & u(N) \end{pmatrix} = \text{new matrix } \underbrace{-\tilde{A}} + \begin{pmatrix} 1/u(1) & & \\ & \ddots & \\ & & \frac{1}{u(N)} \end{pmatrix}$$

punchline: $\boxed{\text{“Weyl law”}}$ ... counting eigenvalues for how many times $k^2 < V^*$

$V^* = 100 \quad k = 1, \ldots, 10$

Filoche Mayboroda used eigenvalue counting to speed up algorithm to simulate performance of LEDs by a factor of 500 to 1000

1 yr → 10 hrs.

Lecture 26

Intro to Fourier Series

4/17/19

- periodic functions
- square waves
- $\boxed{\text{fourier coefficient formula}}$
- orthogonality

Motivation

$$\ddot{x} + A\dot{x} + Bx = f(t)$$

any periodic input

PS 8
sq(ωt)

Def' f(t) is periodic of period p

if $f(t+p) = f(t)$

· what is the period of $\sin(3t)$ ☆ trick question: many answers

Ans: any multiple of $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{6\pi}{3}$, $\frac{8\pi}{3}$ ...

shortest

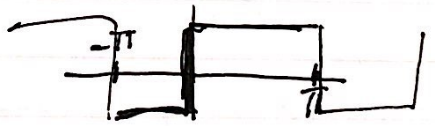
"All" sinusoidal functions w/ period 2π

$\sin t$, $\sin 2t$, $\sin 3t$, ..., $\sin(nt)$ $n = 1, 2, \ldots$

$\cos(nt)$ $n = 0, 1, 2 \ldots$

↑ includes $\cos(0t) = 1$

$$Sq(t) = \begin{cases} +1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$



mathlet

$\frac{1}{c} Sq(t)$

$c = 4/\pi$

$\boxed{2\pi\text{-periodic}}$

to interpret, need to expand in fourier series. How can be written as a superposition of basic building blocks

$$f(t) = c\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right)$$ fourier sine series

Fourier Analysis: split a signal into its pure frequencies.

If $f(t) = \sum_n b_n \sin(nt)$, what are the b_n's?

hard step: finding the coefficients

answer is better than a decimal expansion approximace functions using sine [illegible] waves ← Fourier

What is the 101st digit of $\sqrt{2}$?

tedious to find and essentially meaningless

In contrast b_{101} is easy to find and has a specific meaning.

Fourier $f(t) = \sum b_n \sin(nt)$

$$\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \int_{-\pi}^{\pi} (b_1 \sin t + b_2 \sin 2t + \dots)\sin nt\,dt$$

$$\cancel{\int_{-\pi}^{\pi} (\sin(lt))(\sin(nt))\,dt} \qquad \int_{-\pi}^{\pi} (\sin(mt))(\sin(nt))\,dt = 0 \text{ interested in } m \neq n$$

$$= \int_{-\pi}^{\pi} b_n (\sin(nt))^2\,dt = \pi b_n$$

$$\int_{-\pi}^{\pi} \sin^2(nt)\,dt \qquad \int_{-\pi}^{\pi} (\sin^2 nt + \cos^2 nt)\,dt = \int_{-\pi}^{\pi} 1\,dt = 2\pi$$

$$= \int_{-\pi}^{\pi} \cos^2(nt)\,dt = \pi$$

Fourier coefficient formula

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

a_0 is special because 1... is not oscillatory

a_0 is special because 1 is not oscillatory

"Fourier Theorem" every periodic Function $f(t)$ of period 2π

Satisfies $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$

Every means all piecewise differential functions

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Sq(t)\sin(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} 1 \ \sin nt \ dt$$

both odd
so now even.

$\cos(n\pi) = (-1)^n,$
$\cos 0 = 1$

$$= \frac{-2}{\pi}\left(\frac{\cos(nt)}{n}\right)\Big|_0^{\pi} = \frac{-2}{\pi}\left(\frac{(-1)^n - 1}{n}\right)$$

$$= \begin{cases} 4/\pi n & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

To get the cosines as well as sines, we need several identities

$$\boxed{\int_{-\pi}^{\pi} \sin^2 nt\,dt = \int_{-\pi}^{\pi} \cos^2 nt\,dt = \pi}$$

$$\boxed{\int_{-\pi}^{\pi} (\sin(mt))(\sin(nt))\,dt = 0 \quad m \neq n}$$

$$\boxed{\int_{-\pi}^{\pi} 1^2\,dt = 2\pi \quad \left(\frac{a_0}{2} \text{ term}\right)}$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,1\,dt$$

$=$ <u>average value of f</u>

$$\boxed{\int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt = 0 \quad \text{all } n, m}$$

$\int_0^{2\pi}$ same as $\int_{-\pi}^{\pi}$

Recitation: Farier Series

Taylor Series $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Farier Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$

Find farier series for a 2π periodic funcion

$n \geq 0 \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$n \geq 1 \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$g(-x) = -g(x)$ odd , $g(-x) = g(x)$

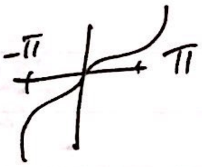
(even)(even) = even

(even)(odd) = odd

(odd)(odd) = even

g odd funcion

$\int_{-\pi}^{\pi} g(x)\,dx = 0$



$= \int_{-\pi}^{0} g(x)\,dx + \int_{0}^{\pi} g(x)\,dx$

cancels out.

since $f(x)$ is odd, $f(x)\cos(nx)$ is odd,

$a = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$

For an odd function, all cosins in the farier series will be zero

For an even funcion, all sine terms in the farier series are zero

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx - \frac{1}{\pi}\int x\sin(nx)\,dx$

$u\,dv \qquad dv = \sin(nx)$

integration by parts

$u = x \qquad v = \frac{-\cos(nx)}{n}$

$du = dx$

$\int u\,dv - uv = \int v\,du$

~~$-\frac{x\cos(nx)}{n}$~~

[illegible] $= uv$

$$-\frac{x\cos(nx)}{n}\Bigg|_{-\pi}^{\pi} - \cancel{\int_{-\pi}^{\pi}\left(\frac{-\cos(nx)}{n}\right)dx}\ 0 \qquad = uv$$

$$\int v\,dx$$

$$\int \ddot{u}\,dv \ \ldots \quad \int_{-\pi}^{\pi} \frac{-\cos(nx)}{n}\,dx = \left[\frac{-\sin(nx)}{n^2}\right]_{-\pi}^{\pi} = 0$$

$$= \frac{1}{\pi}\left(\frac{-2\pi(1)^n}{n} = \frac{-2(-1)^n}{n}\right.$$

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n\sin(nx)}{n} = -2\sin(x) - \frac{2\sin(2x)}{2} + \frac{2\sin(3x)}{3}$$

$$- \frac{2\sin(4x)}{4} \quad \ldots$$

4/19/19 Lecture 27: Working with Fourier Series

- scaling period $2\pi \to 2L$
- odd / even shortcuts
- Convergence

$$Sq(\omega t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(n\omega t)}{n}$$

Scale

PS8

Suppose $g(t)$ period $2L$ $\pi \leftrightarrow L$ $g(t+2L) = g(t)$

building blocks: $\cos(n \frac{\pi}{L} t) \nwarrow$ period $2L$

$n = 0, 1, 2$

$\sin(n \frac{\pi}{L} t) \swarrow$ $n = 1, 2, 3, ..$

Fourier Thm

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

with $a_n = \frac{1}{L} \int_{-L}^{L} g(t) \cos\left(\frac{n\pi}{L} t\right)$, $b_n = \frac{1}{L} \int_{-L}^{L} g(t) \sin\left(\frac{n\pi}{L} t\right)$

Proof:

$f(r) = g\left(\frac{L}{\pi} r\right)$ $t = \frac{L}{\pi} r$ $\boxed{r = \frac{\pi}{L} t}$

$f(r)$ . . . $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(r) \cos(nr)\, dr =$ right answer.

$t = \frac{L}{\pi} r$ $dt = \frac{L}{\pi} dr$

A function f is odd if $f(-t) = -f(t)$

even if $f(-t) = f(t)$

Extra part of fourier coeff. formulas

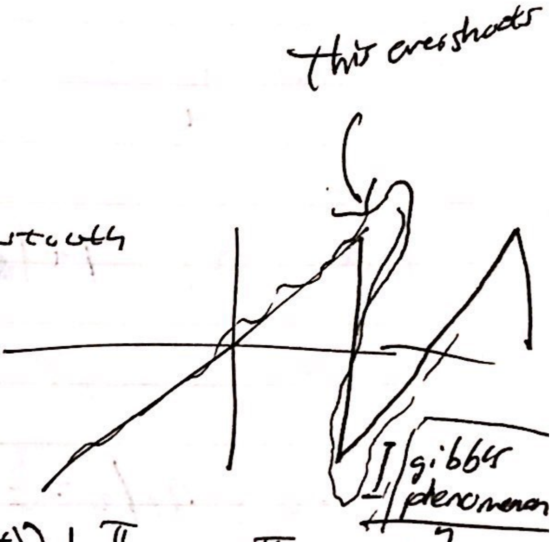
suppose $g(-t) = -g(t)$ odd $\int_0^L = -\int_{-L}^0 \Rightarrow \int_{-L}^0 + \int_0^L = 0$

$$\int_{-L}^{L} \underbrace{g(t)}_{\text{odd}} \underbrace{\cos(n \tfrac{\pi}{L} t)}_{\text{even}}\, dt = 0$$

Hence $a_n = 0$ for all n

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{g(t)}_{odd}\ \underbrace{\sin\left(\frac{n\pi}{L}t\right)}_{odd}}_{even} dt = \frac{2}{L}\int_0^L g(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

$$\int_{-L}^{0} = \int_0^L \qquad \int_{-L}^{L} = 2\int_0^{L}$$



Example $W(t) = t$ Sawtooth

period 2π

$a_n = 0$ because odd

$$b_n = \frac{2}{\pi}\int_0^\pi t\sin nt\,dt$$

$$\int_0^\pi t\sin(nt)\,dt = \left(-\frac{t\cos nt}{n} + \underbrace{\frac{\sin(nt)}{n^2}}_{0-0}\right)\Bigg|_0^\pi = \frac{-\pi}{n}(-1)^n$$

$\cos(n\pi) = (-1)^n$

Don't FORGET

$$\frac{2(-1)^{n+1}}{n}$$

$$\boxed{W(t) = 2\sum(-1)^{n+1}\frac{\sin(nt)}{n} = 2\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \dots\right)}$$

Convergence of F series

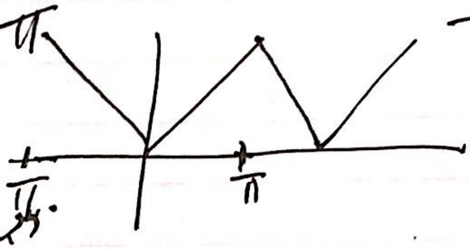
If f is piecewise differentiable with at most jump discontinuities then the Fourier series converges to the value for the average of left ~~hand~~ and right limits.

Even function $g(-t) = g(t)$ $b_n = 0$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(t)\cos\left(\frac{n\pi}{L}t\right)dt = \frac{2}{L}\int_{0}^{L} g(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

$n=0$ $a_0 = \frac{1}{L}\int_{-L}^{L} g(t)dt \Leftrightarrow a_0/2 = \frac{1}{2L}\int_{-L}^{L} g(t)dt$ average value of g

$T(t) = |t|$, $|t| \le \pi$



period 2π "Triangle function"

$b_n = 0$, $a_0/2 = \frac{\pi}{2}$ (average value)

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} t\cos(nt)dt = \begin{cases} \frac{-4}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$ (omitted)

$n = 1, 2, \ldots$

$$T(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\cos nt}{n^2} \quad \boxed{\text{all } t}$$

Lecture 28 : Solving ODEs with Fourier Series 4/22/19

I didn't go ~~to~~ class because I felt sick, so this is what I missed.
(taken Emma lms)

goal for fourier series: solve 2nd order diffeq w/ constant coefficients

undamped

~~[illegible]~~ $x'' + \omega_0^2 x = f(t)$

solve this / find a particular solution.

$$\boxed{f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{\text{odd}} \frac{\sin(\pi t n)}{n}}$$

Can find x_p if RHS: $\begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$ $\quad \dfrac{\begin{cases} \cos \omega t \\ \sin \omega t \end{cases}}{\omega_0^2 - \omega^2}$ } if almost zero, amplitude very large

(use $e^{i\omega t}$)

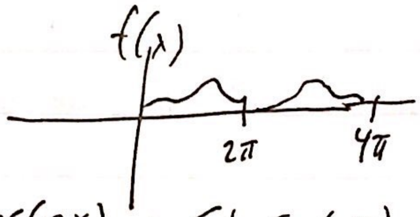
[from lecture on ~~eq~~ resonance]

If $f(t) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t$ $\quad \omega_n = \frac{n\pi}{L}$ (period is $2L$)

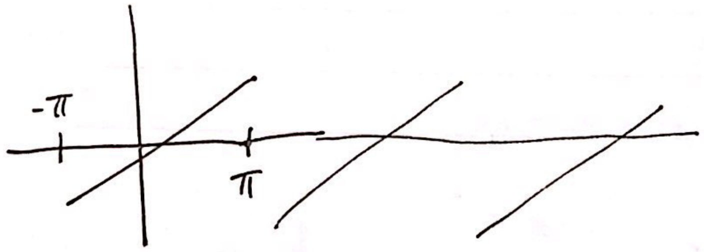
then $x_p = \frac{a_0}{2\omega_0^2} + \sum_1^{\infty} \frac{a_n \cos \omega_n t}{\omega_0^2 - \omega_n^2} + \frac{b_n \sin \omega_n t}{\omega_0^2 - \omega_n^2}$ } particular solution

4/23/19 Fourier Series

$f(x) \Rightarrow$ repeats every 2π ish.



$$f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum b_n\sin(nx)$$



make 2π periodic by repeating it.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)$$

Consider functions which are periodic with period L.

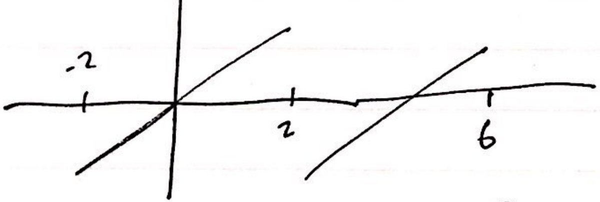
$\sin(\frac{2\pi}{L}nx)$ and $\cos(\frac{2\pi}{L}nx)$ building blocks for Fourier series of functions of period L

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(\frac{2\pi}{L}nx)+\sum_{n=1}^{\infty} b_n\sin(\frac{2\pi}{L}nx)$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos(\frac{2\pi}{L})dx \qquad b_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{2\pi}{L}nx)dx$$

average of the function

$g(x) = x$ between -2 and 2 and is 4-periodic



$$f(x)=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)$$

need to transform to $g(x)$

stretches the function

need to put down → $\frac{2}{\pi}f(\frac{\pi}{2}x) = g(x)$

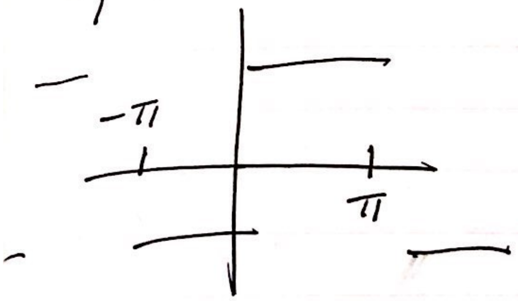
$$\underset{\underset{g(x)}{\shortparallel}}{\frac{2}{\pi} f\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}(-2) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{\pi n}{2} x\right)$$

$\sin\left(\frac{2\pi}{L} \cancel{n} x\right)$ $\cos\left(\frac{2\pi}{L} n x\right)$ $L=4$

$\sin\left(\frac{\pi}{2} n x\right)$ $\sin\left(\frac{\pi}{2} n x\right)$

Find fourier series: find relationship between your function and something you already know.

$Sq(t)$



2π periodic

$$Sq(t) = \frac{4}{\pi} \sum_{\substack{K=1 \\ K=odd}}^{\infty} \frac{\sin(Kt)}{K}$$

$$\ddot{x} + 2\dot{x} + 2 = Sq(t)$$

$$\ddot{x} + 2\dot{x} + 2 = \frac{4}{\pi} \sum_{K=odd} \frac{\sin(Kt)}{K}$$

$\downarrow$

$$\ddot{x} + 2\dot{x} + 2 = \frac{4}{\pi} \frac{\sin(Kt)}{K} \longrightarrow \ddot{\tilde{x}} + 2\dot{\tilde{x}} + 2 = \frac{4}{4} \frac{e^{iKt}}{K}$$

$\cancel{r^2+2r+2=0}$

$r = -1 \pm i$

$p(D)x$ where $p(D) = D^2 + 2D + 2$

$$\tilde{x} = \frac{4}{\pi n} \frac{e^{ikt}}{p(ik)} \qquad \tilde{x} = \frac{4}{\pi k} \frac{e^{ikt}}{(-k^2+2) + 2ik}$$

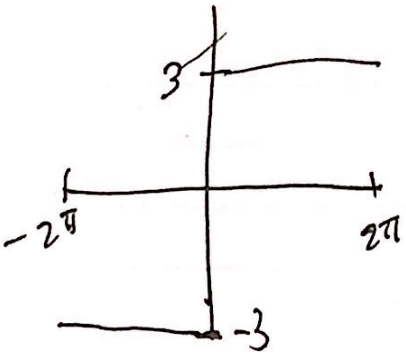
$$\frac{4}{\pi k((-k+2)^2 + 4k^2)} (2\cos(kt) + (2-k^2)\sin(kt))$$

$\cancel{x(t) = \frac{4}{\pi} \sum_{K odd} \frac{1}{k(1}}$

$$= \frac{8}{\pi} \sum_{K odd} \frac{1}{k((-k+2)^2 + 4k^2)} \cos t$$

$$+ \frac{4}{\pi} \sum_{K odd} \frac{2-k^2}{k((k+2)^2 + 4k^2)} \sin t$$

Let $g(x)$ be the 4π periodic function which is 3 from 0 to 2π and -3 from -2π to 0.



What is its 4π peridic Farier Series?

29-1,2
30-1,2

Lecture 29: Intro to Partial Differential Equations 4/24/19

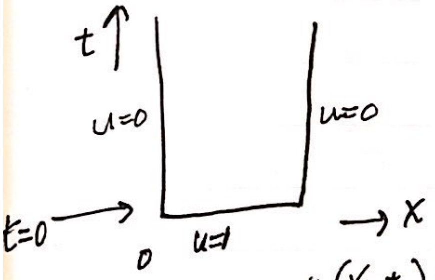
- the heat equation
- separation of variables
- boundary value problems

Hint on PS8: try narrower window to get more resonant frequencies

heat equation describes the temperature

$0 \leq x \leq L$
metal bar

initially a temperature 1, ends are plunged into 0. See how rod cools.



space-time
(x, t)

two independent variables

$u(x,t) = $ temp at (x,t)

Heat equation $\frac{\partial u}{\partial t} = r\frac{\partial^2 u}{\partial x^2}$ r conductivity constant (diffusion constant)

boundary conditions

$u(x,0)=1$ $0 \leq x \leq L$ || $u(0,t)=0$ $t>0$ (left) $u(L,t)=0$ $t>0$ (right)

$u(L/2, t)$

$\sim ce^{-t/\tau}$ $(t \to \infty)$

newton's law of cooling

copper choose unit of time $T=1$ || iron $T \sim 8$ || concrete $T \sim 200$

T inversely proportional to the r

Also c is universal $c \sim 1.3$

$u(x,t) \sim \frac{4}{\pi}\sin(\pi/L\, x)e^{-t/T}$ $t > T$

sin is the solution to differential equation, which is why it comes up so much.

x fixed $x = \frac{L}{2} \Rightarrow \frac{\pi x}{L} = \frac{\pi}{2}$ $\boxed{c = \frac{4}{\pi}}$

Separation of Variables

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ (set r=1 for simplicity) (set $L=\pi$ for simplicity)

$0 \leq x \leq \pi$ ~~$u(x,0)=1$~~

get rid of this condition

$u(0,t)=0$
$=u(\pi,t)$ $t>0$

Fourier's tricks

- Abandon one of the conditions: the initial one
- Separation of variables: $u(x,t) = v(x)w(t)$ ← look first for these solutions

Plug into the PDE

$$v(x)\dot{w}(t) = \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}u = v''(x)w(t)$$

$$\frac{\dot{w}(t)}{w(t)} = \frac{v''(x)}{v(x)} \Rightarrow \text{implies both sides are constants} \quad \frac{\dot{w}}{w} = \lambda = \frac{v''(x)}{v(x)} \quad \underline{\lambda \text{ constant}}$$

$$\frac{\dot{w}}{w} = \lambda \quad \dot{w} = \lambda w$$

$$w(t) = ae^{\lambda t} \rightsquigarrow \lambda = -\frac{1}{T}$$

$$\frac{v''(x)}{v(x)} = \lambda \iff \boxed{v'' = \lambda v} \quad 0 \le x \le L$$

boundary conditions $\boxed{v(0) = v(\pi) = 0}$

Case 0 $\lambda = 0 \quad v'' = 0 \quad v(x) = ax + b \quad v(0) = v(\pi) = 0 \rightarrow a\pi = 0 \Rightarrow a = 0$

gives only the boring solution: $v = 0$ $\quad \rightarrow$ forces $b = 0$

Case 1 $\lambda > 0 \quad v(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \quad v(0) = 0 \Rightarrow c_1 + c_2 = 0$

$v \equiv 0 \quad v(x) = c(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) \quad v(\pi) = 0 \Rightarrow c(e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi})$

Case 2 $\lambda < 0 \quad \omega = \sqrt{-\lambda} > 0$

solutions span $\cos(\omega x), \sin(\omega x) \quad \omega^2 = \lambda$

$v(x) = a\cos(\omega x) + b\sin(\omega x) \quad v(0) = 0 \Rightarrow 0 = a\overbrace{\cos 0}^{1} = a$

Lastly $v(\pi) = 0 \Rightarrow b\sin(\omega\pi) = 0$

Hence we need $\omega = 1, 2, 3, 4 \ldots$

Solutions $v_n(x) = \sin(nx)$, ~~span~~ $\lambda = -n^2$

eigenfunction eigenvalue

solution (normal modes)

$$w_n(t)v_n(x)$$

$$\boxed{e^{-n^2 t}\sin(nx)}$$

now we need to ~~go back to~~ $u(x,0)=1$ (the boundary condition we dropped)

$\boxed{A\vec{v}=\lambda\vec{v}}$ analogy

Important for quantum mechanics

you get discrete numbers as opposed to continuous (?)

$V(x) \longleftrightarrow$ [illegible] $\vec{v}=\begin{pmatrix} v(x_1) \\ v(x_2) \\ \vdots \\ v(x_n) \end{pmatrix}$ $\quad x_k=\frac{k}{N}\pi \quad 1\le k\le N$

function

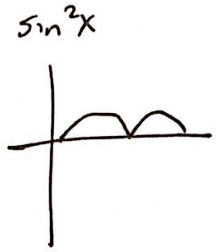
$\frac{d^2}{dx^2}V=\lambda V$ $\qquad A\vec{v}=\lambda\vec{v}$

Recitation

4/25/19

Trick:

$\sin^2 x$



Compute the integral

$$\int_{-\pi}^{\pi} \sin^2(x)\,dx$$

$$\cos(x) = \sin\left(x+\frac{\pi}{2}\right)$$

$$\cos^2(x) = \sin^2\left(x+\frac{\pi}{2}\right)$$

$$\int_{-\pi}^{\pi} \cos^2(x)\,dx = \int_{-\pi}^{\pi} \sin^2\left(x+\frac{\pi}{2}\right)$$

$$u = x+\frac{\pi}{2}$$

$$= \int_{-\pi+\frac{\pi}{2}}^{\pi+\frac{\pi}{2}} \sin^2(u)\,du$$

$$\int_{-\pi/2}^{\pi} \sin^2(u)\,du + \int_{-\pi}^{\pi/2} \sin^2(u)\,du = \int_{\pi} \sin^2(u)\,du \qquad = \int_{-\pi/2}^{\pi} \sin^2 u\,du + \int_{\pi}^{3\pi/2} \sin^2(u)\,du$$

$$2\int_{-\pi}^{\pi} \sin^2 x\,dx = \int_{-\pi}^{\pi} \sin^2(x)\,dx + \int_{-\pi}^{\pi} \cos^2(x)\,dx = \int_{-\pi}^{\pi} \left[\sin^2(x) + \cos^2(x)\right]dx$$

$$= \int_{-\pi}^{\pi} 1\,dx = 2\pi \quad \Rightarrow \int_{-\pi}^{\pi} \sin^2(x)\,dx = \pi$$

$\ddot{x} + 4x =$ ~~cos~~ $\sin(2t)$ $\qquad x = im(\tilde{x}) \quad \sin(2t) = im(e^{2it})$

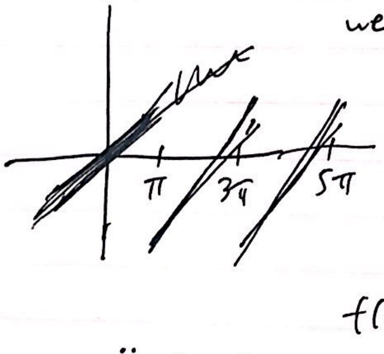
$\ddot{\tilde{x}} + 4\tilde{x} = e^{2it}$

$P(D) = D^2 + 4$

$$\boxed{\begin{array}{l} \underline{e^{2it}} \\ p(2i) \leftarrow \text{gets } 0\text{, so we need to do generalized erf} \end{array}}$$

$\Rightarrow$ since erf doesn work, the solutio is unbounded. we call this <u>resonance</u>

$f(x) = x$ when x is between $-\pi$ and π we want to make it 2π periodic.



$\Leftarrow$ this is an odd function $f(-x) = -f(x)$

so we have sines in the furier serie.

$$f(x) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$

$$\ddot{x} + 4x = f(x)$$

$$\ddot{x} + 4x = \frac{-2(-1)^n}{n} \sin(nx)$$

$\searrow$ solutions to this differential equation are unbounded

$x^2+10x=f(x)$ resonance frequency $= i\sqrt{10}$.

no resonance
because $f(x)$'s terms ~~resonance frequency~~ are integers, so there is no $\sqrt{10}$